# LONGITUDINAL MOTION

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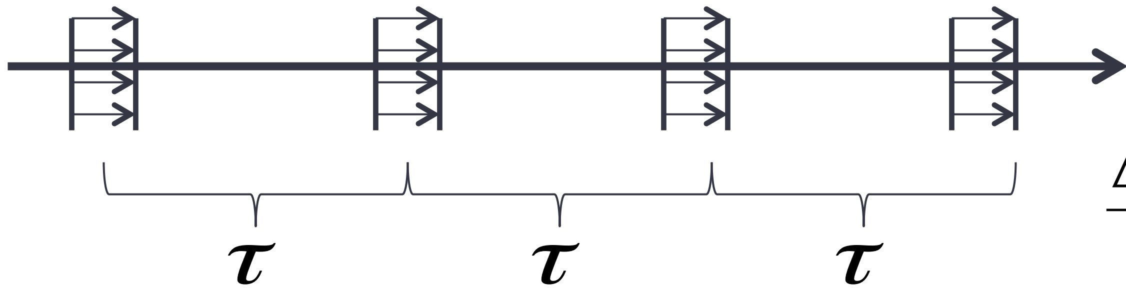
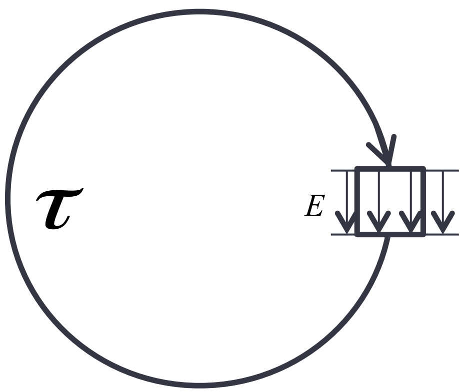
# Acceleration in Periodic Structures

- We consider motion of particles either through a linear structure or in a circular ring




Always negative

$$\frac{\Delta\tau}{\tau} = -\frac{1}{\gamma^2}\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$




In both cases, we can adjust the RF phases such that a particle of nominal energy arrives at the the same point in the cycle $\phi_\sigma$

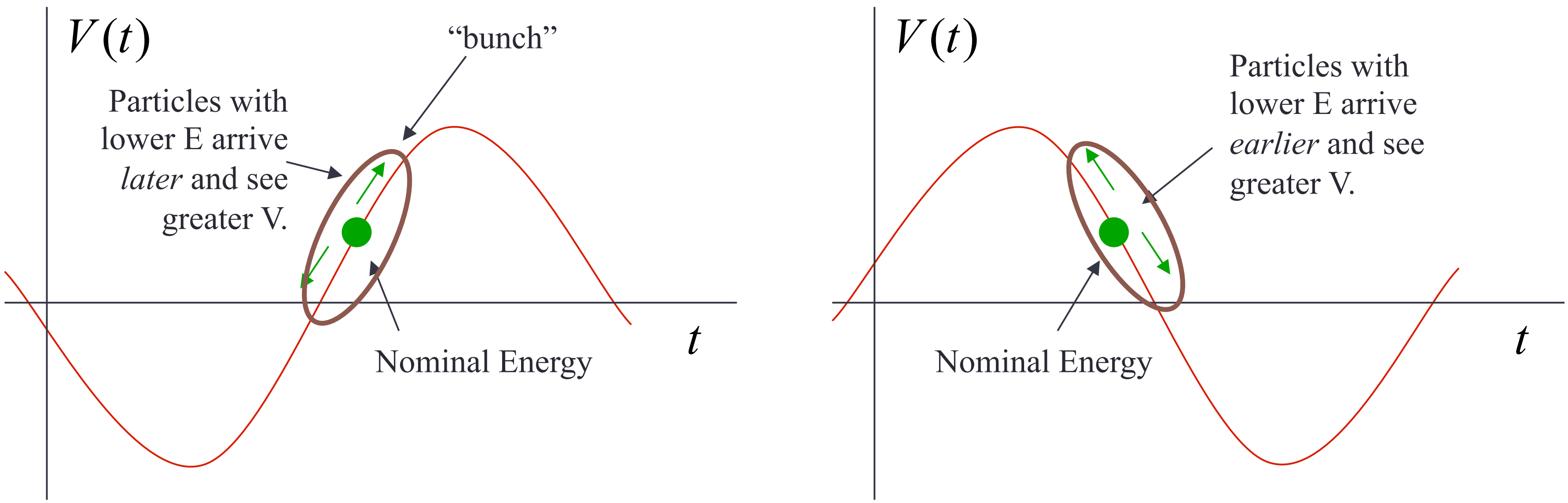
$$\frac{\Delta\tau}{\tau} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$

Goes from negative to positive at transition

# Slip Factors and Phase Stability

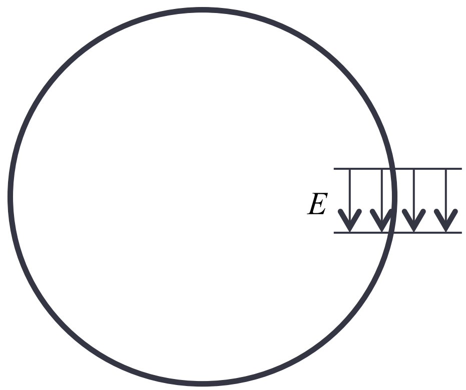
- The sign of the slip factor determines the stable region on the RF curve.

η<0 (linacs and below transition) η>0 (above transition)



# Longitudinal Acceleration

- Consider a particle circulating around a ring, which passes through a resonant accelerating structure each turn




$$V = V_0 \sin\left(\omega_{RF} t + \omega_0\right); \quad \omega_{RF} = 2\pi \frac{h}{\tau}$$

Harmonic number (number of RF cycles per period) → $h$

Period of nominal energy particle → $\tau$

- The energy gain that a particle of the nominal energy experiences each turn is given by

$$E_{n+1} = E_n + eV_0 \sin\phi_s$$

← Synchronous phase

where this phase will be the same for a particle on each turn

- A particle with a different energy will have a different phase, which will evolve each turn as

$$\phi_{n+1} = \phi_n + \Delta\phi$$

← Phase difference from one turn to the next

# Phase Difference

- The phase difference can be expressed as

$$\Delta\phi = \omega_{RF}\Delta t = \omega_{RF}\tau\eta\frac{\Delta p}{p} = 2\pi h\frac{1}{\beta^2}\frac{\Delta E}{E}$$

use $\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{E_S\beta^2}\Delta E$$

"synchronous" (nominal) energy

# Longitudinal Equation of Motion

- Thus the change in energy for this particle for this particle will evolve as

$$\Delta E_{n+1} = \Delta E_n + eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

Constant (or very slowly varying)

- So we can write

$$\frac{d\phi}{dn} = \frac{2\pi h\eta}{E_s\beta^2}\Delta E$$

$$\frac{d\Delta E}{dn} = eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

$$\Rightarrow \frac{d^2\phi}{dn^2} = \frac{eV_0 2\pi h\eta}{E_s\beta^2}\left(\sin\phi_n - \sin\phi_s\right)$$

exact

# Synchrotron Motion and Synchrotron Tune

- Rewrite this equation as:

$$\frac{d^2\phi}{dn^2}+\left(-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\left(\sin\phi_n-\sin\phi_s\right)\right)=0$$

- For *small* oscillations,

$$\sin\phi_n-\sin\phi_s\approx\left.\frac{d\sin\phi}{d\phi}\right|_{\phi=\phi_s}\left(\phi_n-\phi_s\right)=\Delta\phi\cos\phi_s$$

- And we have

$$\frac{d^2\Delta\phi}{dn^2}+\left(-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s\right)\Delta\phi=0$$

- This is the equation of a harmonic oscillator with

$$\omega_n=\sqrt{-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s}\Rightarrow\nu_s=\frac{1}{2\pi}\sqrt{-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s}$$

Angular frequency wrt *turn* (not time)

"synchrotron tune" = number of oscillations per turn (usually <<1)

# Longitudinal Phase Space and Emittance

- We want to write things in terms of time and energy. We have can write the longitudinal equations of motion as

$$\Delta t(n) = \frac{1}{\omega_{rf}} \Delta\phi(n)$$

$$\frac{d\Delta t(n)}{dn} = \frac{1}{\omega_{rf}} \frac{d\Delta\phi(n)}{dn} = \frac{\tau\eta}{E_S\beta^2} \Delta E(n)$$

- Following our procedure for longitudinal motion, we want to write this in form:

$$\Delta t(n) = a\cos(2\pi\nu_s n) + b\sin(2\pi\nu_s n)$$

$$\Delta t(0) \equiv \Delta t_0 = a \rightarrow a = \Delta t_0$$

$$\left.\frac{d\Delta t(n)}{dn}\right|_{n=0} = 2\pi\nu_s b = \frac{\tau\eta}{E_S\beta^2}\Delta E_0 \rightarrow b = \frac{\tau\eta}{2\pi E_S\beta^2\nu_s}$$

$$\Delta t(n) = \Delta t_0 \cos\left(2\pi\nu_s n\right) + \frac{\tau\eta}{2\pi E_S\beta^2\nu_s} \Delta E_0 \sin\left(2\pi\nu_s n\right)$$

- Taking the derivative wrt *n* and substituting for $\Delta E$ gives us

$$\Delta E(n) = \frac{E_S\beta^2}{\tau\eta}\frac{d\Delta t(n)}{dn} = \Delta E_0 \cos\left(2\pi\nu_s n\right) - \frac{2\pi E_S\beta^2\nu_s}{\tau\eta}\Delta t_0 \sin\left(2\pi\nu_s n\right)$$

- So we can write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos\left(2\pi\nu_s n\right) & \dfrac{\tau\eta}{2\pi E_S\beta^2\nu_s}\sin\left(2\pi\nu_s n\right) \\ -\dfrac{2\pi E_S\beta^2\nu_s}{\tau\eta}\sin\left(2\pi\nu_s n\right) & \cos\left(2\pi\nu_s n\right) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- But we've seen that before!

- This looks just like our equation for transverse motion with α=0, so we immediately write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_s n) & \beta_L \sin(2\pi\nu_s n) \\ -\gamma_L \sin(2\pi\nu_s n) & \cos(2\pi\nu_s n) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- Where

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \frac{\tau|\eta|}{2\pi E_S\beta^2\nu_s} = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\varphi_s}};\gamma_L = \frac{1}{\beta_L}$$

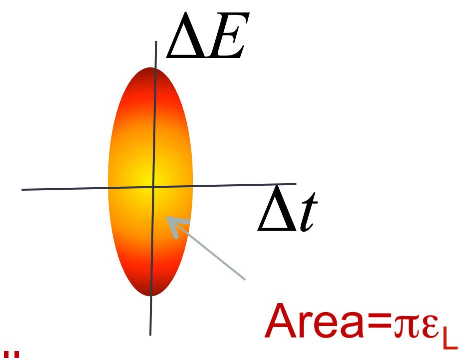
Units: s/eV

Units: eV/s

- We can define an invariant of the motion as

$$\frac{1}{\beta_L}(\Delta t)^2 + \beta_L(\Delta E)^2 \equiv \epsilon_L = \text{constant}$$

units generally eV-s




- What about the behavior of $\Delta t$ and $\Delta E$ separately?

$$\Delta E_{RMS} = \sqrt{\frac{\epsilon_L}{\beta_L}} = \left(-\frac{\epsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta}\right)^{\frac{1}{4}}$$

$$\Delta t_{RMS} = \sqrt{\epsilon_L \beta_L} = \left(-\frac{\epsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}\right)^{\frac{1}{4}}$$

$$\Rightarrow \frac{\Delta t}{\Delta E} = \beta_L$$

Aspect ratio

- Note that for linacs or well-below transition

$$\eta = -\frac{1}{\gamma^2} \Rightarrow \Delta E \propto \left(\frac{E}{1/\gamma^2}\beta^2\right)^{\frac{1}{4}} \propto \left(\gamma^3\beta^2\right)^{\frac{1}{4}}; \Delta t \propto \left(\gamma^3\beta^2\right)^{-\frac{1}{4}}$$

# Large Amplitude Motion

- We have solved for the synchrotron tune in the limit of small oscillations, but in general we will not restrict ourselves to small oscillations.
- Recall our exact equations of motion:

$$\frac{d\phi}{dn}=\frac{2\pi h\eta}{E_s\beta^2}\Delta E$$

$$\frac{d\Delta E}{dn}=eV_0\left(\sin\phi_n-\sin\phi_s\right)$$

$$\Rightarrow\frac{d^2\phi}{dn^2}=\frac{eV_0 2\pi h\eta}{E_s\beta^2}\left(\sin\phi_n-\sin\phi_s\right)$$

substitute

- Multiply both sides by $\frac{d\phi}{dn}$ and integrate over *dn*

constant

$$\int\left(\frac{d\phi}{dn}\frac{d^2\phi}{dn^2}\right)dn=\frac{eV_0 2\pi h\eta}{E_s\beta^2}\int\left(\sin\phi(n)-\sin\phi_s\right)\frac{d\phi}{dn}dn$$

$$\Rightarrow\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2=-\frac{eV_0 2\pi h\eta}{E_s\beta^2}\left(\cos\phi+\phi\sin\phi_s\right)+\text{constant}$$

extrema

$$\Rightarrow\frac{1}{2}\left(\Delta E\right)^2+\frac{eV_0E_s\beta^2}{2\pi h\eta}\left(\cos\phi+\phi\sin\phi_s\right)=\frac{eV_0E_s\beta^2}{2\pi h\eta}\left(\cos\phi_0+\phi_0\sin\phi_s\right)$$

exact

# Orbits

- The relationship between phase angle and ΔE is given by

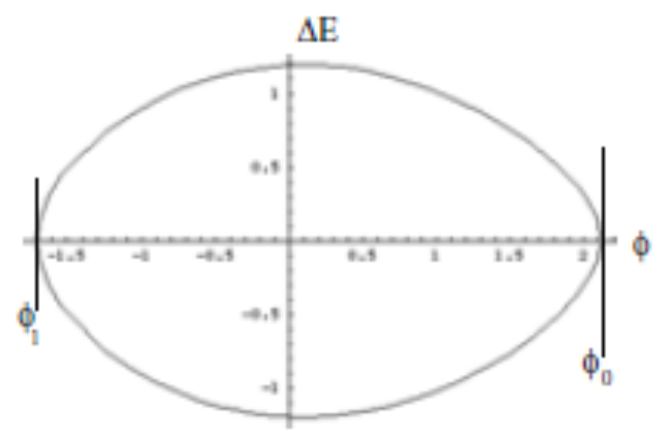
$$\frac{1}{2}(\Delta E)^2 + \frac{eV_0E_s\beta^2}{2\pi h\eta}(\cos\phi + \phi\sin\phi_s) = \frac{eV_0E_s\beta^2}{2\pi h\eta}(\cos\phi_0 + \phi_0\sin\phi_s)$$

$$\frac{1}{2}(\Delta E)^2 + \frac{eV_0E_s\beta^2}{\omega_{rf}\tau\eta}(\cos\phi + \phi\sin\phi_s) = \frac{eV_0E_s\beta^2}{\omega_{rf}\tau\eta}(\cos\phi_0 + \phi_0\sin\phi_s)$$

$$\frac{1}{2}(\Delta E)^2 + \frac{-1}{\omega_{rf}^2\beta_L^2\cos\phi_s}(\cos\phi + \phi\sin\phi_s) = \frac{-1}{\omega_{rf}^2\beta_L^2\cos\phi_s}(\cos\phi_0 + \phi_0\sin\phi_s)$$

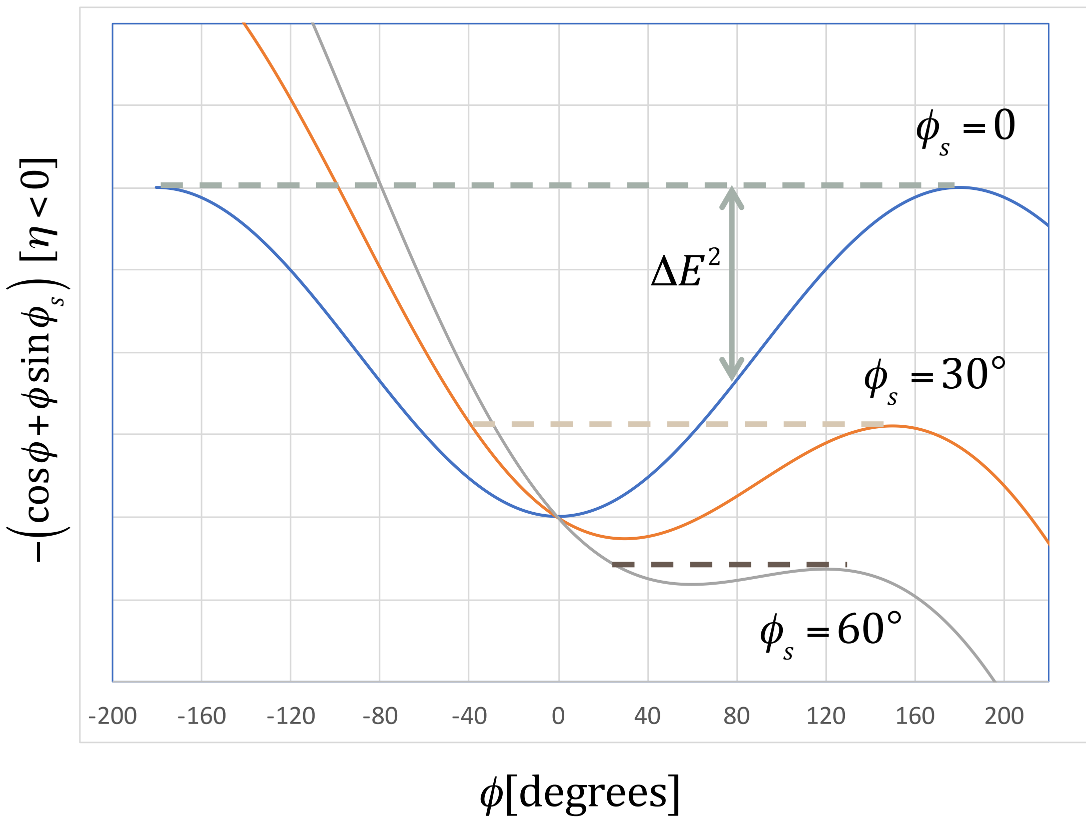
Use:

$$\beta_L^2 = -\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}$$



# Stable Region

- The range in $\phi$ will be determined by the classical turning points




One end point of the maximum stable orbit is defined by the right maximum

$$\frac{d}{d\phi}\left(-\cos\phi - \phi\sin\phi_s\right) = +\sin\phi - \sin\phi_s$$
$$= 0$$

$$\sin\phi = \sin\phi_s$$
$$\phi = \phi_s \rightarrow \text{minimum}$$
$$\phi = \pi - \phi_s \rightarrow \text{maximum}$$

The other end is defined by

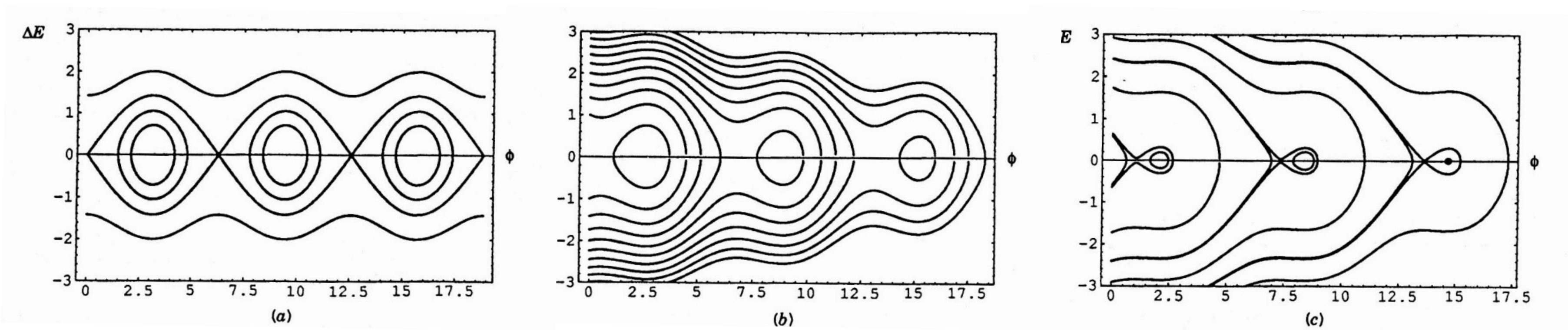
$$\cos\phi_1 + \phi_1\sin\phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s$$
$$= -\cos\phi_s + (\pi - \phi_s)\sin\phi_s$$

# Separatrix

- The "separatrix", or maximum stable orbit, is defined by

$$\frac{1}{2}(\Delta E)^2 + \frac{-1}{\omega_{rf}^2 \cos\phi_s \beta_L^2}\left(\cos\phi + \phi\sin\phi_s\right) = \frac{-1}{\omega_{rf}^2 \cos\phi_s \beta_L^2}\left(\cos(\pi-\phi_s) + \phi_s(\pi-\phi_s)\sin\phi_s\right)$$

$$\frac{1}{2}(\Delta E)^2 = \frac{\left(\phi\sin\phi_s + \cos\phi\right) - \left((\pi-\phi_s)\sin\phi_s - \cos\phi_s\right)}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$




$\phi_s = 0$ $\phi_s = 30°$ $\phi_s = 60°$

# Longitudinal Bucket

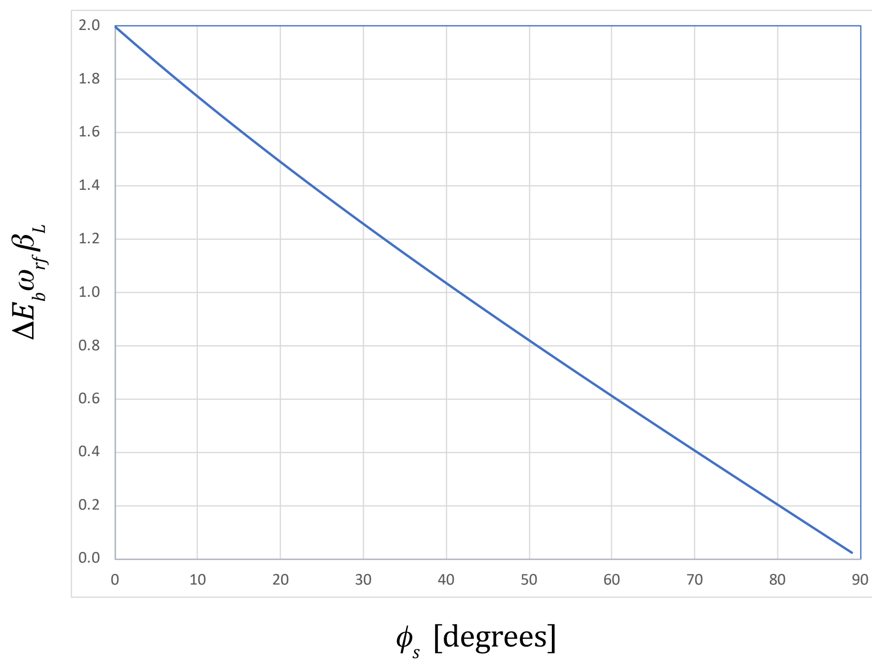
- The "bucket" is the area contained within the separatrix
- The "bucket height" is the maximum ΔE, which will occur when $\phi=\phi_s$.

$$(\Delta E)^2 = 2\frac{[\cos\phi + \phi\sin\phi_s] + [\cos\phi_s - (\pi - \phi_s)\sin\phi_s]}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$

$$(\Delta E_b)^2 = 2\frac{2\cos\phi_s + 2\phi_s\sin\phi_s - \pi\sin\phi_s}{\omega_{rf}^2\cos\phi_s\beta_L^2}$$

$$= \frac{4}{\omega_{rf}^2\beta_L^2}\left(1 - \left(\frac{\pi}{2} - \phi_s\right)\tan\phi_s\right)$$

$$\Delta E_b = 2\frac{\sqrt{1 - \left(\frac{\pi}{2} - \phi_s\right)\tan\phi_s}}{\omega_{rf}\beta_L}$$



# Transition Crossing

- We learned that for a simple FODO lattice $\gamma_T \approx \nu$ so electron machines are always above transition.
- Proton machines are often designed to accelerate through transition.
- As we go through transition $(\eta < 0) \Rightarrow (\eta = 0) \Rightarrow (\eta > 0)$
- Recall

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}} = \frac{\Delta t_{\max}}{\Delta E_{\max}}$$

At transition:

$$\Delta t_{\max} \Rightarrow \text{constant}$$

$$\Delta E_{\max} \Rightarrow \infty$$

so these both go to zero at transition.

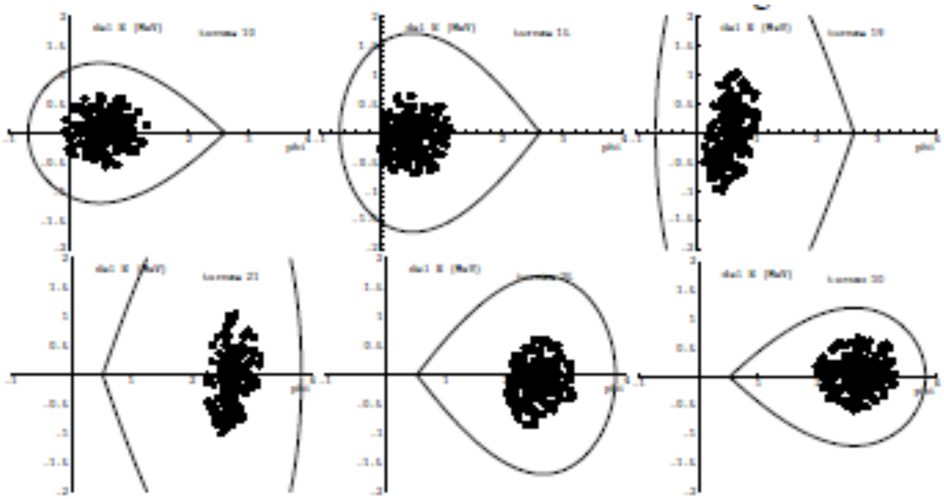
- To keep motion stable

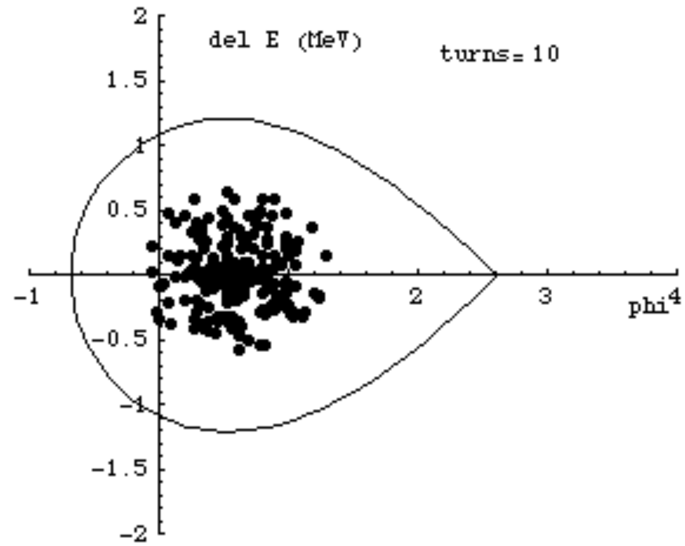
$$\cos\phi_s > 0 \text{ below transition;} \Rightarrow 0 < \phi_s < \frac{\pi}{2}$$

$$\cos\phi_s < 0 \text{ above transition;} \Rightarrow \frac{\pi}{2} < \phi_s < \pi$$

# Effects at Transition

- As the beam goes through transition, the stable phase must change*



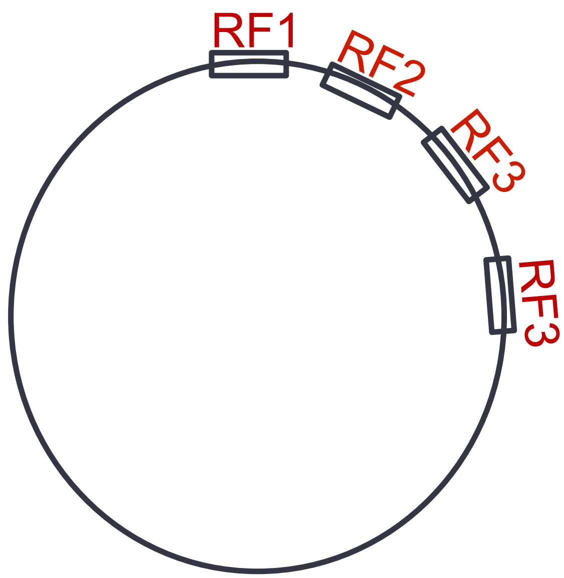



- Problems at transition (pretty thorough treatment in S&E 2.2.3)
  - Beam loss at high dispersion points
  - Emittance growth due to non-linear effects
  - Increased sensitivity to instablities
  - Complicated RF manipulations near transition
    - Much harder before digital electronics

*animations from Gerry Dugan

# RF Manipulations

- As you'll show in homework, the synchrotron tune (longitudinal oscillations/ turn) is generally <<1.
- That means that if there are multiple RF cavities around the ring, the orbiting particle will see the *vector sum* of the cavities.




$$\frac{\Delta E}{dn} = \sum_{i=1}^{N} V_i \sin(\phi_i)$$
$$= V_{eff} \sin\left(\phi_{eff}\right)$$

$\phi_i$ is the phase angle at the arrival of the particle at cavity *i*

- We will clearly get the maximum energy gain if all phases are the same, so (assuming all voltages are the same)

$$\frac{\Delta E}{dn} = NV_0 \sin\left(\phi_s\right)$$

# Do we always want the maximum acceleration?

- As we'll see, there are times when we want to change the amplitude of the RF quickly.
- Because cavities represent stored energy, changing their amplitude quickly can be difficult.
- Much quicker to change phase
- Standard technique is to divide RF cavities into two groups and adjust the relative phase. In the simplest case, we put half the RF cavities into group "A" and half into group "B". We can adjust the phases of these cavities relative to our nominal synchronous phase as

$$
\begin{aligned}
V_{eff}\sin\left(\phi_{eff}\right) &= \frac{N}{2}V_0\sin\left(\phi_s+\delta\right)+\frac{N}{2}V_0\sin\left(\phi_s-\delta\right)\\
&= \frac{N}{2}\left(\sin\phi_s\cos\delta+\cos\phi_s\sin\delta+\sin\phi_s\cos\delta-\cos\phi_s\sin\delta\right)\\
&= NV_0\cos\delta\sin\phi_s
\end{aligned}
$$

- So $V_{eff} = NV_0\cos\delta;\phi_{eff} = \phi_s$

$$\delta = 0 \Rightarrow V_{eff} = NV_0$$

$$\delta = \frac{\pi}{2} \Rightarrow V_{eff} = 0$$

Like "turning RF off"

# Adiabatic Capture*

- We initially capture the beam by raising the RF voltage “adiabatically” (over many synchrotron oscillations). This insures that the longitudinal phase space stays matched to the RF bucket

dp/p

z [m]

z [m]

- If instead we simply turn the RF on, the beam will “filament”.

dp/p

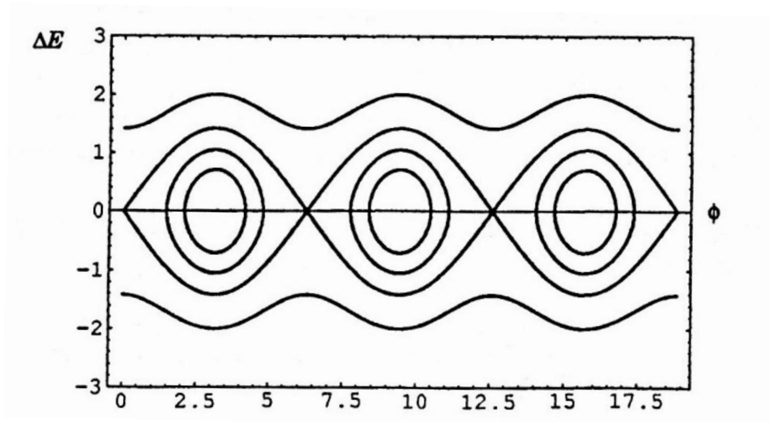
z [m]

z [m]

(*Synergia simulations: IOTA ring, protons, K=2.5 MeV, τ=1.77 μsec, h=4, $V_{max}$=1kV)

# Capture and Acceleration

- We can adiabatically capture beam by increasing the RF voltage with no accelerating phase




- As we accelerate beam, Δt decreases. Recall $\frac{1}{\beta_L}(\Delta t)^2 + \beta_L(\Delta E)^2 \equiv \epsilon_L = \text{constant}$

$$\Delta E_{RMS} = \sqrt{\frac{\varepsilon_L}{\beta_L}} = \left(-\frac{\varepsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta}\right)^{\frac{1}{4}}$$

$$\Delta t_{RMS} = \sqrt{\varepsilon_L \beta_L} = \left(-\frac{\varepsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}\right)^{\frac{1}{4}}$$

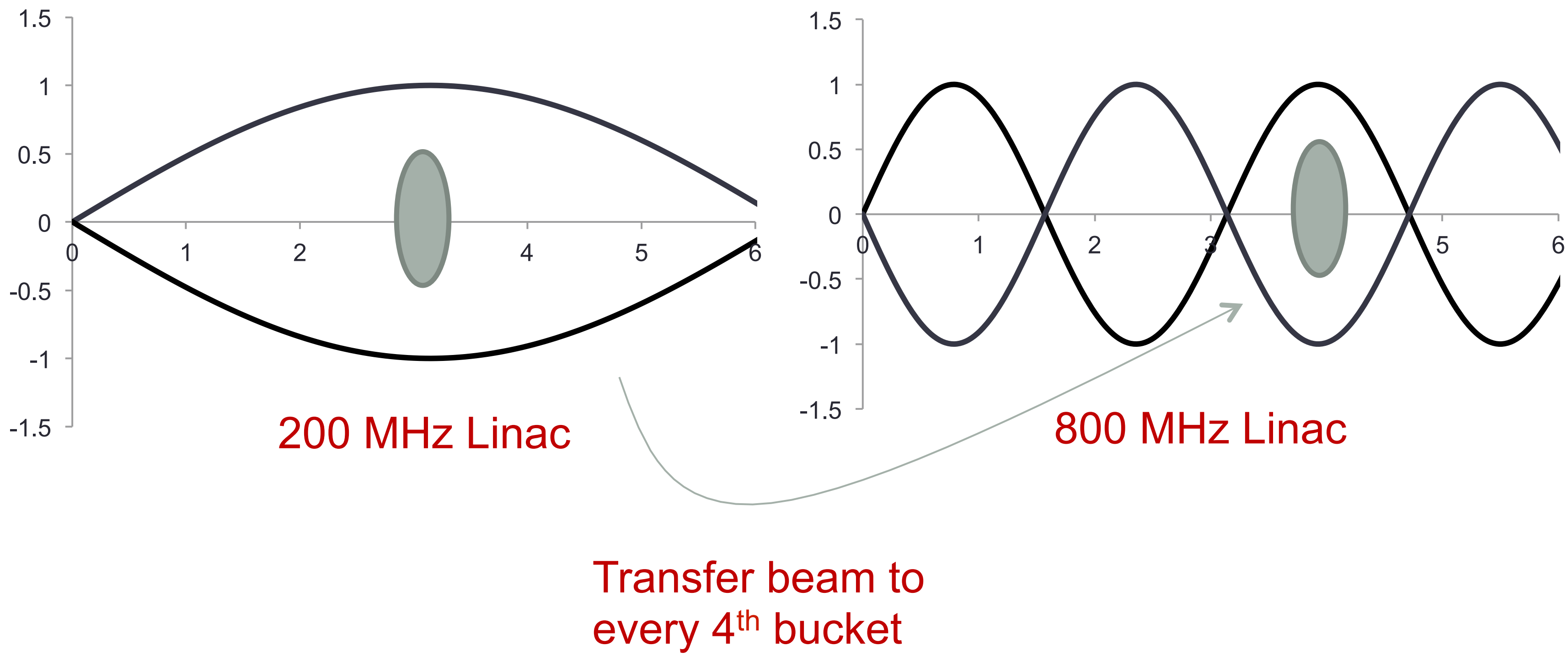
- So as beam accelerates, bunches get narrower

# Bucket to Bucket Transfer

- In general, the accelerating gradient of an RF structure is

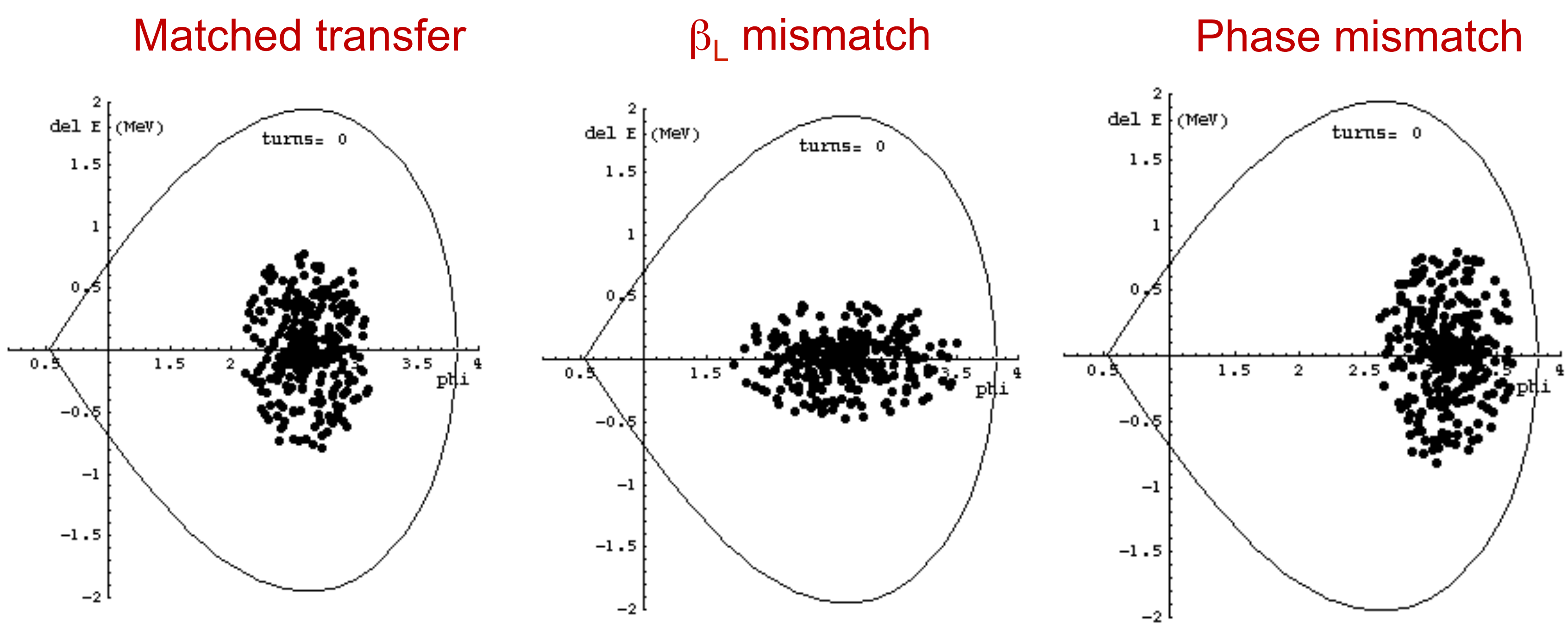
$$\frac{V}{L} \propto \frac{V_{breakdown}}{\lambda_{RF}} \propto \omega_{RF} V_{breakdown}$$

- So when bunches get short enough, it's advantageous to transfer to a higher frequency. For example, in the Fermilab Linac



# Bucket Transfers: Phase and Beta Matching

- When we transfer beam from one machine to another, or from a lower frequency section to a higher frequency section, it's important to correctly match the phases!



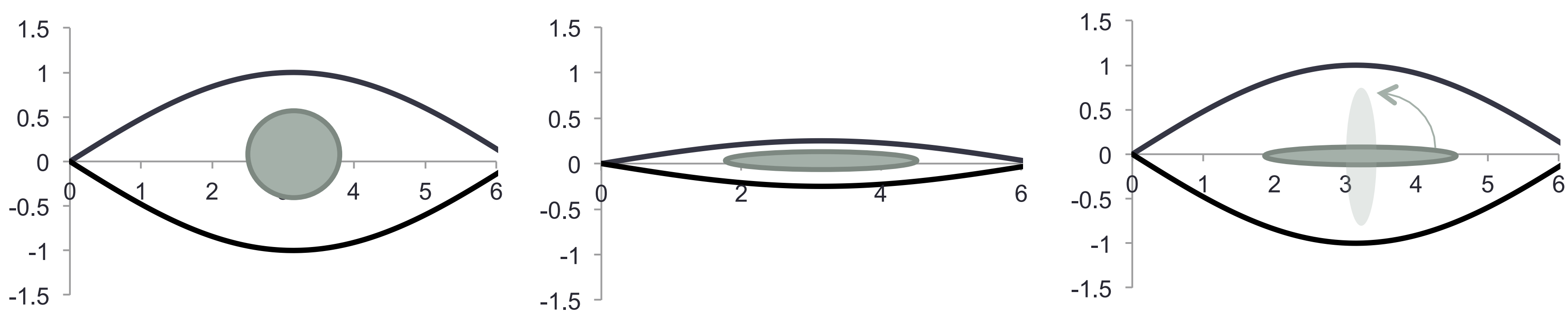
# Bunch Rotation

- If we *slowly* change the RF voltage (or effective voltage by phasing), we can adiabatically change the bunch shape

$$\Delta E_{RMS} = \sqrt{\frac{\varepsilon_L}{\beta_L}} = \left( -\frac{\varepsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta} \right)^{\frac{1}{4}} \propto V_0^{\frac{1}{4}}$$

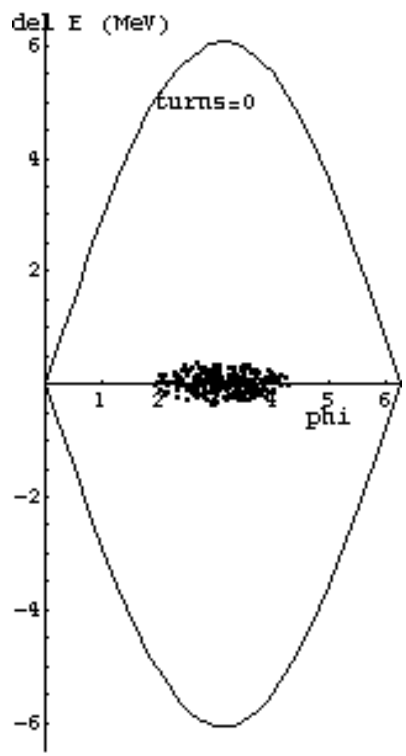
$$\Delta t_{RMS} = \sqrt{\varepsilon_L \beta_L} = \left( -\frac{\varepsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s} \right)^{\frac{1}{4}} \propto V_0^{-\frac{1}{4}}$$

- If we suddenly change the voltage, then the bunch will be mismatched and will rotate in longitudinal phase space



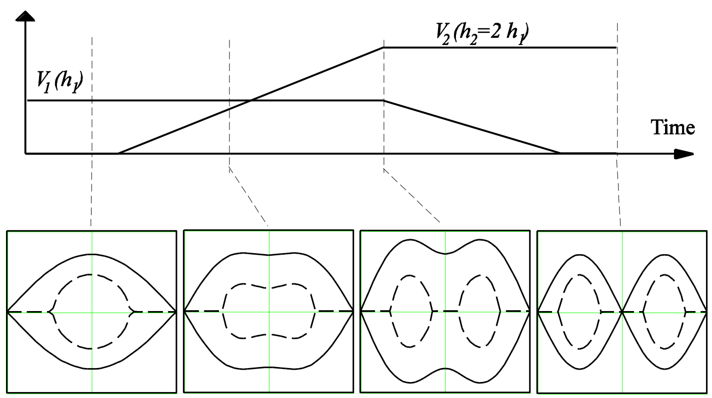
# Bunch Rotation (cont'd)

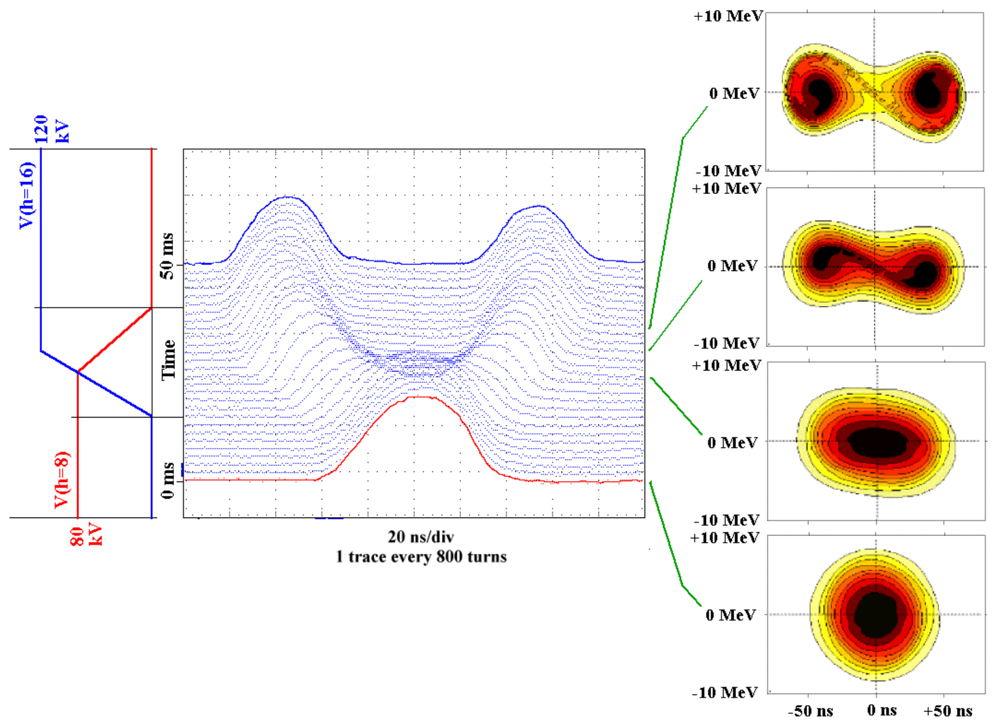
- Of course, non-adiabatically increasing the RF voltage ("snapping") will cause the beam to filament, but the effect is minor over ¼ of a synchrotron oscillation



# RF "Gymnastics"

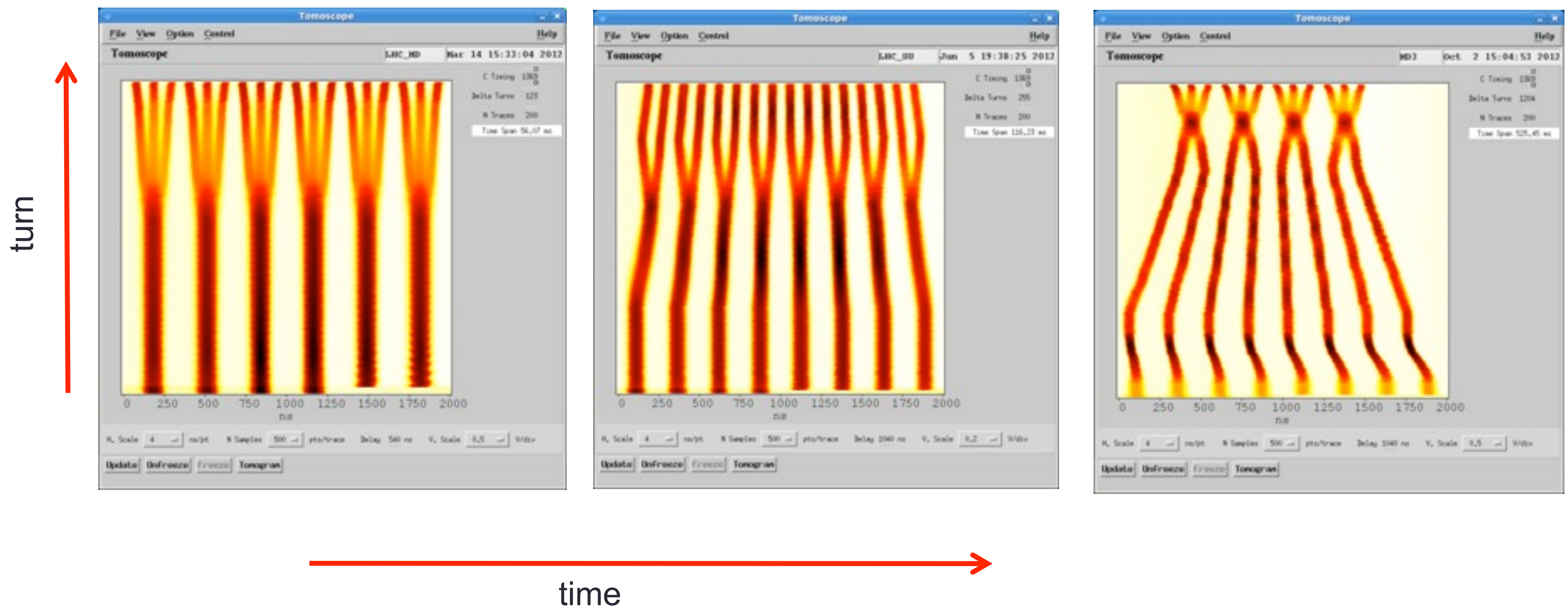
- By manipulating the RF in the system, very sophisticated things can be done*
  - Example: Bunch splitting:




*R. Garoby, "RF Gymnastics in Synchrotrons"

- Example: Bunch formation in CERN PS for LHC*




*S. Hancock