OFF-MOMENTUM PARTICLES

Eric Prebys, UC Davis

Off-Momentum Particles

- Our previous discussion implicitly assumed that all particles were at the same momentum
  - Each quad has a constant focal length
  - There is a single nominal trajectory
- In practice, this is never true. Particles will have a distribution about the nominal momentum
- We will characterize the behavior of off-momentum particles in the following ways
  - "Dispersion" (*D*): the dependence of position on deviations from the nominal momentum υ_x

$$\Delta x(s) = D_x(s)\frac{\Delta p}{p_0}$$

D has units of length

  - "Chromaticity" (η) : the change in the tune caused by the different focal lengths for off-momentum particles

$$\Delta \upsilon_x = \xi_x \frac{\Delta p}{p_0} \quad \left(\text{sometimes} \frac{\Delta \upsilon_x}{\upsilon_x} = \xi_x \frac{\Delta p}{p_0} \right)$$

  - Path length changes (momentum compaction)

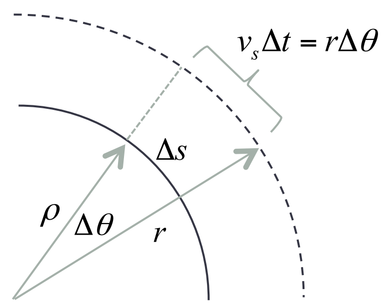
$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

Review: Equations of Motion

- Recall that in a curvilinear coordinate system, the equations of motion become

$$x'' = -\frac{B_y}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho+x}{\rho^2}$$

$$y'' = \frac{B_x}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2$$



Note: *s* measured along *nominal* trajectory, v_s measured along *actual* trajectory

$$\dot{s} = \frac{ds}{dt} = \frac{\rho}{r}v_s$$

- We'll now consider consider the effect of of off momentum particle by comparing the "true" rigidity to the nominal rigidity

$$(B\rho)_{true} = (B\rho)\frac{p}{p_0} \rightarrow \frac{1}{(B\rho)_{true}} = \frac{1}{(B\rho)}\frac{p_0}{p} = \frac{1}{(B\rho)}\frac{p_0}{(p_0+\Delta p)} \approx \frac{1}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)$$

Off-Momentum Particles

- If we substitute this into the equations of motion, and keep only linear terms, we end up with one new term in each equation

$$x'' = -\frac{B_y}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho+x}{\rho^2} = (\ldots) + \frac{B_y}{(B\rho)}\frac{\Delta p}{p_0} \equiv (\ldots) + \frac{B_y}{(B\rho)}\delta$$

$$y'' = \frac{B_x}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)\left(1+\frac{x}{\rho}\right)^2 = (\ldots) - \frac{B_x}{(B\rho)}\delta$$

- The parts in parentheses just give us our nominal equations of motion. We now invoke

$$B_x = B'y \approx 0$$

$$B_y = B_0 + B'x \approx B_0;$$

$$\Rightarrow \quad \frac{B_y}{(B\rho)} \approx \frac{B_0}{(B\rho)} = \frac{1}{\rho}$$

$$\frac{B_x}{(B\rho)} \approx 0$$

- And our new equations become

$$x'' + \left(\frac{1}{\rho^2} + \frac{1}{(B\rho)}B'\right)x = \boxed{\frac{1}{\rho}\delta;} \qquad y'' - \frac{1}{(B\rho)}B'y = 0$$

New

- This is a second order differential inhomogeneous differential equation, so the solution is

$$x(s) = x_0 C(s) + x_0' S(s) + \delta d(s)$$
$$x'(s) = x_0 C'(s) + x_0' S'(s) + \delta d'(s)$$

Where d(s) is the solution particular solution of the differential equation

$$d'' + Kd = \frac{1}{\rho}$$

- We solve this piecewise, for *K* constant and find

$$K > 0: \quad d(s) = \frac{1}{\rho K}\left(1 - \cos\sqrt{K}s\right)$$
$$d'(s) = \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s$$
$$K < 0: \quad d(s) = -\frac{1}{\rho K}\left(1 - \cosh\sqrt{K}s\right)$$
$$d'(s) = \frac{1}{\rho\sqrt{K}}\sinh\sqrt{K}s$$

General Solution

- The general solution is now

$$x(s) = x_0 C(s) + x_0' S(s) + \delta d(s)$$
$$x'(s) = x_0 C'(s) + x_0' S'(s) + \delta d'(s)$$

Solution to the on-momentum case

Off-momentum correction

- We can express this in matrix form as

Usual transfer matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & d(s) \\ m_{21} & m_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

New Equilibrium Orbit

- We want to solve for an orbit of an off-momentum particle that follows the periodicity of the machine.
- This will serve as the new equilibrium orbit for off-momentum particles.

"Dispersion" [L]

$$x(s,\delta) = D_x(s)\delta$$

- This must satisfy

$$\begin{pmatrix} \delta D_x \\ \delta D'_x \\ \delta \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & d \\ M_{21} & M_{22} & d' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta D_x \\ \delta D'_x \\ \delta \end{pmatrix} \Rightarrow \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} = (\ldots) \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix}$$

Simplifying Assumptions

- For the most part, we will consider systems for which both of the following are true
  - "separated function": Separate dipoles and quadrupoles

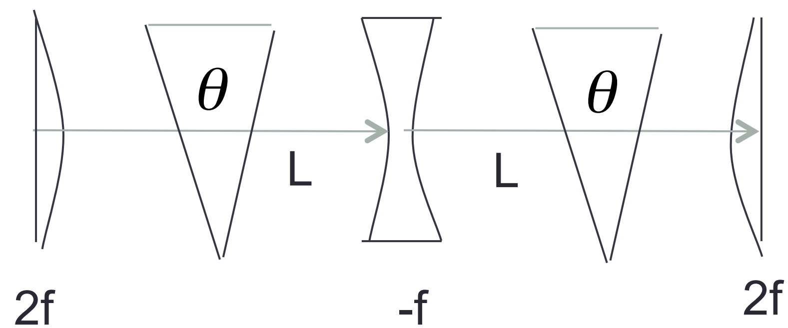
$$\Rightarrow \frac{1}{\rho^2} \text{ and } B' \text{ are never both non-zero at the same point}$$

  - "Isomagnetic": All bend dipoles have the same field

$$\begin{aligned} \frac{1}{\rho^2} &= \frac{1}{\rho_0^2} \quad \text{inside of bend dipoles} \\ &= 0 \quad \text{everywhere else} \end{aligned}$$

Example: FODO Cell

- We look at our symmetric FODO cell, but assume that the drifts are bend magnets that take up the entire space (a pretty good assumption)
- Each bends the beam by an angle θ



For a thin lens *d~d'~0*. For a pure bend magnet

$s << \rho_0$

$$K = \frac{1}{\rho_0^2}: \quad d(L) = \frac{1}{\rho_0 K}\left(1 - \cos\sqrt{K}L\right) = \rho_0\left(1 - \cos\frac{L}{\rho_0}\right) \approx \frac{1}{2\rho_0}L^2 \rightarrow \frac{1}{2}\frac{L^2}{\rho_0} = \frac{1}{2}\theta L$$

$$d'(L) = \frac{1}{\rho_0\sqrt{K}}\sin\sqrt{K}L = \sin\frac{L}{\rho_0} \approx \frac{L}{\rho_0} \rightarrow \theta$$

Transfer Matrix

- We put this all together to get a transfer matrix of the form

Usual transfer matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & d(s) \\ M_{21} & M_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

- Using our solutions from the previous page, we get

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L\left(1+\frac{L}{2f}\right) & 2L\theta\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta\left(1-\frac{L}{4f}-\frac{L^2}{8f^2}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

- For a ring:

$$\theta = \frac{2\pi}{2N_{cell}} = \frac{\pi}{N_{cell}}$$

Solving for Dispersion

- We must solve

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L\left(1+\frac{L}{2f}\right) & 2L\theta\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta\left(1-\frac{L}{4f}-\frac{L^2}{8f^2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

- In your homework, you show that

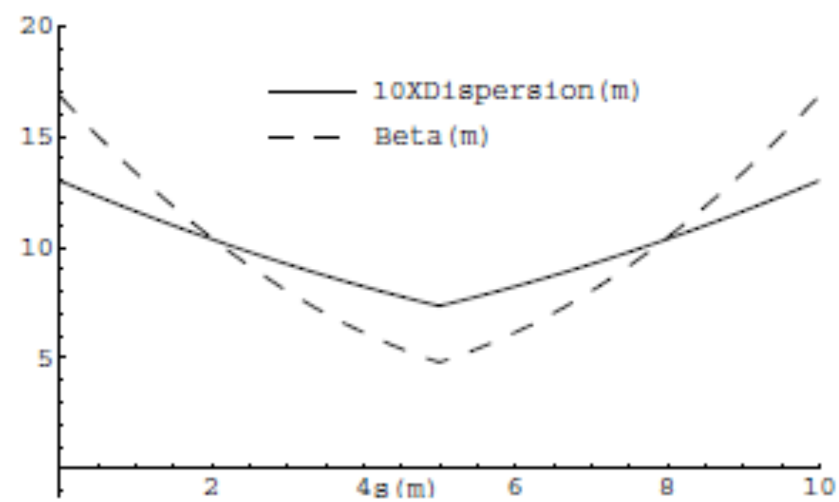
$$D_{F,D} = \frac{\theta L\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

Evolution of Dispersion Functions

- Since the dispersion functions represent displacements, they will evolve like the position

$$\begin{pmatrix} D_x(s) \\ D'_x(s) \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & d(s) \\ m_{21} & m_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_x(0) \\ D'_x(0) \\ 1 \end{pmatrix}$$

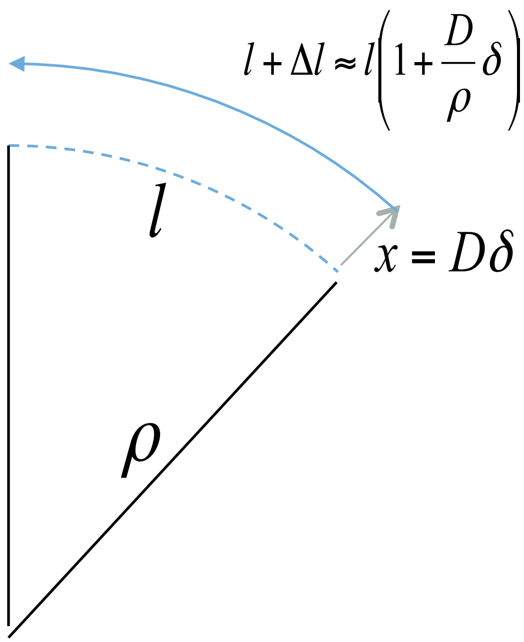
- Putting it all together



Momentum Compaction Factor

- In general, particles with a high momentum will travel a longer path length. We have

$$C(p_0) = \oint ds$$



$$C(p_0 + \Delta p) = \oint \left(1 + \frac{D}{\rho}\frac{\Delta p}{p_0}\right) ds$$

$$\frac{\Delta C}{C} = \frac{\oint \frac{D}{\rho} ds}{\oint ds}\delta = \left\langle \frac{D}{\rho} \right\rangle \delta$$

"momentum compaction" factor

$$\equiv \alpha_c \delta$$

So yes, we now have an ambiguous definition of α, too!

Slip Factor

- The "slip factor" is defined as the fractional change in the orbital period divided by the fractional change in momentum

$$T = \frac{C}{v}$$

$$\frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = \frac{\Delta C}{C} - \frac{\Delta \beta}{\beta}$$

$$= \alpha \frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$= \left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

$$\equiv \eta \frac{\Delta p}{p}$$

$$\gamma < \frac{1}{\sqrt{\alpha}}: \quad \eta < 0 \quad \text{velocity dominates}$$

$$\gamma > \frac{1}{\sqrt{\alpha}}: \quad \eta > 0 \quad \text{momentum dominates}$$

$$\gamma = \frac{1}{\sqrt{\alpha}}: \quad \eta = 0 \quad \text{"transition"}$$

Transition gamma or "gamma-T" →

$$\gamma_T \equiv \frac{1}{\sqrt{\alpha}} \Rightarrow \eta = \left(\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}\right)$$

Special Cases for Slip Factor

- Linacs:

$$\alpha = 0 \rightarrow \eta = -\frac{1}{\gamma^2} \text{ (always negative)}$$

- Simple Cyclotrons:

$$C = 2\pi\rho = 2\pi\frac{p}{eB} \rightarrow \alpha = 1 \rightarrow \eta = \left(1 - \frac{1}{\gamma^2}\right) \text{ (0 to positive)}$$

- Synchrotrons: more complicated
  - Negative below γ_T
  - Positive above γ_T

$$\eta = \left(\alpha - \frac{1}{\gamma^2}\right)$$

Transition γ for Synchrotrons (approx.)

- For a simple FODO CELL

$$\beta_{\max,\min} = 2L\frac{\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu}; \text{ and } D_{\max,\min} = \theta L\frac{\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

- If we assume they vary ~linearly between maxima, then for small μ

$$\langle\beta\rangle \approx \frac{2L}{\mu};\ \langle D\rangle \approx \frac{4\theta L}{\mu^2} = 4\frac{L^2}{\mu^2\rho} = \frac{\langle\beta\rangle^2}{\rho}$$

- Also

$$\nu = \frac{1}{2\pi}\oint\frac{ds}{\beta(s)} \approx \frac{1}{2\pi}\frac{2\pi R}{\langle\beta\rangle} \approx \frac{\rho}{\langle\beta\rangle}$$

(cont'd)

- We just showed

$$\langle D\rangle \approx \frac{\langle \beta\rangle^2}{\rho}$$

$$\nu \approx \frac{\rho}{\langle \beta\rangle}$$

$$\Rightarrow \quad \langle D\rangle \approx \frac{\rho}{\nu^2}$$

- So

$$\alpha_c = \frac{1}{C}\oint \frac{D}{\rho} ds \approx \frac{1}{\rho}\langle D\rangle \approx \frac{1}{\nu^2}$$

$$\boxed{\gamma_t = \frac{1}{\sqrt{\alpha_c}} \approx \nu}$$

- This approximation generally works better than it should
  - FNAL Booster: ν=6.8, γ_T=5.5

Digression: Quadrupole Perturbation

- We can express the matrix for a complete revolution at a point as

$$\mathbf{M}(s) = \begin{pmatrix} \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu & \beta(s)\sin 2\pi\nu \\ -\gamma(s)\sin 2\pi\nu & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu \end{pmatrix}$$

- If we add focusing quad at this point, we have

$$\mathbf{M}'(s) = \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 - \frac{\beta(s)}{f}\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 - \frac{1}{f}\left(\cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0\right) & \cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}$$

- We calculate the trace to find the new tune

$$\cos 2\pi\nu = \frac{1}{2}Tr\left(\mathbf{M}'\right) = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$$

- For small changes $\cos 2\pi\left(\nu_0 + \Delta\nu\right) \approx \cos 2\pi\nu_0 - 2\pi\sin 2\pi\nu_0\Delta\nu = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$

$$\Rightarrow \Delta\nu = \frac{1}{4\pi}\frac{\beta(s)}{f}$$

Total Tune Shift

- The focal length associated with a local anomalous gradient is

$$d\left(\frac{1}{f}\right) = \frac{B'}{(B\rho)}ds$$

- So the total tune shift is

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)\frac{B'(s)}{(B\rho)}ds$$

Chromaticity

- In general, momentum changes will lead to a tune shift by changing the effective focal lengths of the magnets
- We already showed

$$\frac{1}{f}=\frac{B'l}{(B\rho)}=\frac{B'l}{(B\rho)_0}\frac{p_0}{p}\approx\frac{1}{f_0}\left(1-\frac{\Delta p}{p_0}\right)$$

$$\Rightarrow\Delta\nu=-\frac{1}{4\pi}\sum_i\beta_i\frac{1}{f_i}\frac{\Delta p}{p_0}\equiv\xi\frac{\Delta p}{p_0}$$

- Where

$$\frac{1}{f_0}=-\int_0^L\frac{B'}{(B\rho)}ds$$

Chromaticity (Cont'd)

- Recalling that in our general equation of motion

$$x'' + \left(\frac{1}{\rho^2} + \frac{B'(s)}{(B\rho)} \right) x = 0 \equiv x'' + K(s)x$$

- We see that the effective focal length for a region is

$$\frac{1}{f_0} = \int_0^L \frac{B'}{(B\rho)} ds \Rightarrow \frac{1}{f_{eff}} = \int_0^L \left(\frac{1}{\rho^2} + \frac{B'}{(B\rho)} \right) ds = \int_0^L K(s) ds$$

- And we can write our general expression for the chromaticity as

$$\xi = -\frac{1}{4\pi} \sum_i \beta_i \frac{1}{f_i} \Rightarrow \xi = -\frac{1}{4\pi} \oint \beta(s) K(s) ds$$

Chromaticity in Terms of Lattice Functions

- A *long* time ago, we derived the following constraint when solving our Hill's equation

$$\beta(s) = w^2(s)$$
$$\alpha(s) = -\frac{1}{2}\beta'(s)$$
$$-\alpha^2 + \beta\gamma = 1$$

$$w''(s) + K(s)w(s) - \frac{k}{w^3(s)} = 0 \Rightarrow \left(\sqrt{\beta}\right)'' + K\sqrt{\beta} - \frac{1}{\beta^{3/2}} = 0$$

$$\left(\sqrt{\beta}\right)' = \frac{1}{2}\frac{1}{\sqrt{\beta}}\beta' = -\frac{\alpha}{\sqrt{\beta}}$$

$$\left(\sqrt{\beta}\right)'' = -\frac{\alpha'}{\sqrt{\beta}} + \frac{1}{2}\frac{\alpha}{\beta^{3/2}}\beta' = -\frac{\alpha'}{\sqrt{\beta}} - \frac{\alpha^2}{\beta^{3/2}}$$

Multiply by $\beta^{3/2}$

$$\Rightarrow K\beta^2 - \beta\alpha' - \alpha^2 = 1$$

- (We're going to use that in a few lectures), but for now, divide by β to get

$$K\beta = \frac{1+\alpha^2}{\beta} + \alpha' = \gamma + \alpha'$$

- So our general expression for chromaticity becomes

$$\xi = -\frac{1}{4\pi}\oint\left(\gamma(s) + \alpha'(s)\right)ds$$

Chromaticity and Sextupoles

- we can write the field of a sextupole magnet as

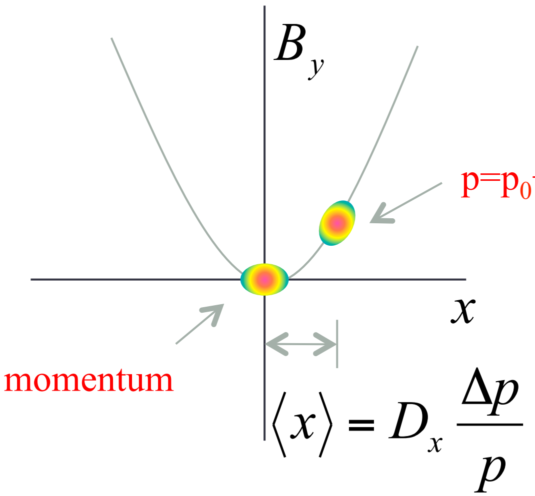
$$B(x) = \frac{1}{2}B''x^2 \qquad \left(\text{often expressed } b_2 x^2\right)$$

- If we put a sextupole in a dispersive region then off momentum particles will see a gradient

$$B'\left(x = D\delta\right) \approx B''D\frac{\Delta p}{p_0}$$

which is effectively like a position dependent quadrupole, with a focal length given by

$$\frac{1}{f_{eff}} = \frac{B''}{(B\rho)}LD\frac{\Delta p}{p_0}$$



- So we write down the tune-shift as

$$\Delta\nu = \frac{1}{4\pi}\beta\frac{1}{f_{eff}} = \frac{1}{4\pi}\frac{\beta B''}{(B\rho)}LD\frac{\Delta p}{p_0} \equiv \xi\frac{\Delta p}{p_0}$$

$$\Rightarrow \xi_S = \frac{1}{4\pi}\frac{\beta B''}{(B\rho)}LD$$

- Note, this is only valid when the motion due to momentum is large compared to the particle spread.