TRANSVERSE MOTION

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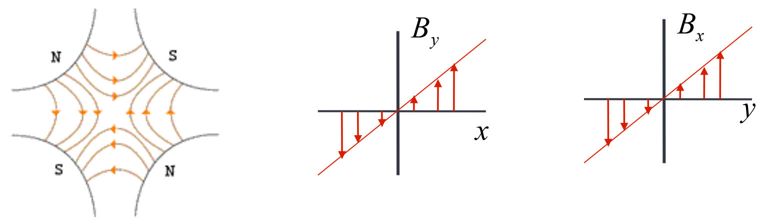
The Journey Begins…

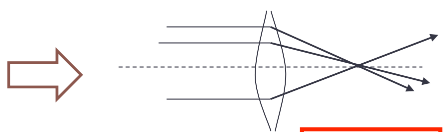
- We will tackle accelerator physics the way we tackle most problems in classical physics – ie, with 18th and 19th century mathematics!
- As we discussed in our last lecture, the linear term in the expansion of the magnetic field is associated with the quadrupole, so let's start there…

Quadrupole Magnets*



- A positive particle coming out of the page off center in the horizontal plane will experience a *restoring* kick

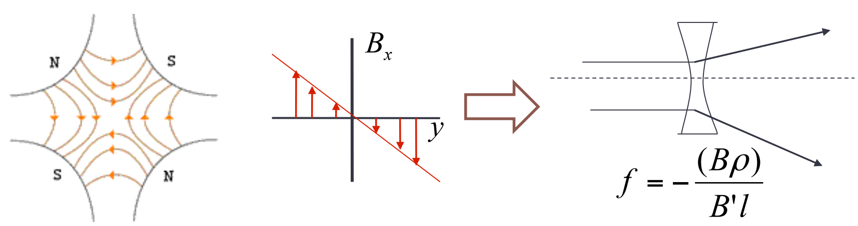
$$\Delta\theta \approx -\frac{B_x(x)l}{(B\rho)} = -\frac{B'lx}{(B\rho)}$$



$$f = \frac{(B\rho)}{B'l}$$

*or quadrupole term in a gradient magnet

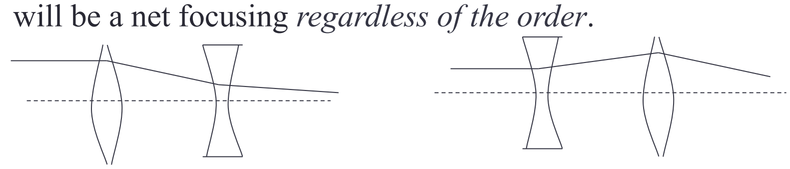
What about the other plane?



$$f = -\frac{(B\rho)}{B'l}$$

Defocusing!

Luckily, if we place equal and opposite pairs of lenses, there will be a net focusing *regardless of the order*.



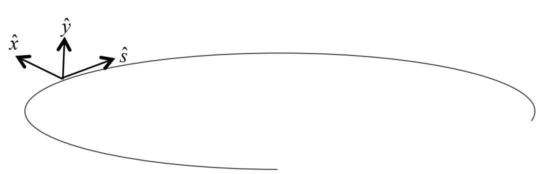
➔pairs give net focusing in *both* planes -> “FODO cell”

General Approach

- The dipole fields in our beam line will define an ideal trajectory.
- The position along this trajectory (*s*) will serve as the independent coordinate of our system.
- We will derive and explicit solution for equations of motion for *linear* focusing or defocusing effect due to deviation from this idea trajectory
  - Linear field gradients (quadrupole term)
  - Curvilinear coordinate system (centripetal term)
- Everything else will be treated as a perturbation to this explicit solution

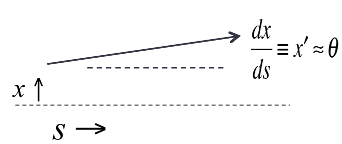
Formalism: Coordinates and Conventions

- We generally work in a right-handed coordinate system with *x* horizontal, *y* vertical, and *s* along the *nominal* trajectory (x=y=0).



Note: *s* (rather than *t)* is the independent variable

- Define derivatives wrt *s*

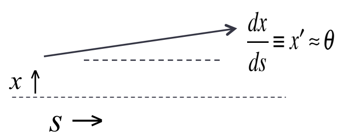


Initial Conditions and Phase Space

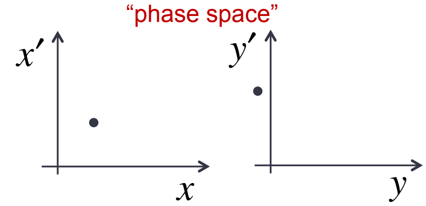
- Our general equations of motion will have the form

$$\frac{d^2x}{ds^2} + K_x(s)x = 0$$
$$\frac{d^2y}{ds^2} + K_y(s)y = 0$$

- These are 2nd order linear homogenous equations, so we need two initial conditions to fully determine the motion



"phase space"

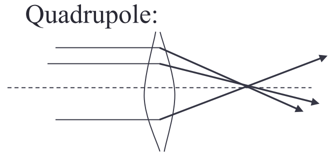


unique initial phase space point ➔ unique trajectory

Transfer matrices

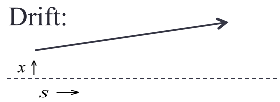
- The simplest magnetic lattice consists of quadrupoles and the spaces in between them (drifts). We can express each of these as a linear operation in phase space.

Quadrupole:



$$x = x(0)$$
$$x' = x'(0) - \frac{1}{f}x(0) \quad \Rightarrow \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

Drift:



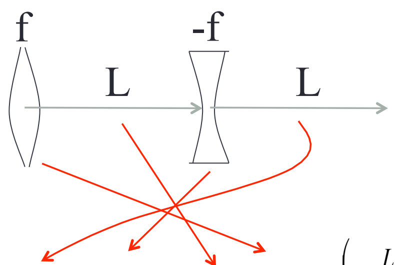
$$x(s) = x(0) + sx'(0)$$
$$x'(s) = x'(0) \quad \Rightarrow \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

- By combining these elements, we can represent an arbitrarily complex ring or line as the product of matrices.

$$\mathbf{M} = \mathbf{M}_N \ldots \mathbf{M}_2 \mathbf{M}_1$$

Example: Transfer Matrix of a FODO cell

- At the heart of every beam line or ring is the basic "FODO" cell, consisting of a focusing and a defocusing element, separated by drifts:



Remember: motion is usually drawn from left to right, but matrices act from right to left!

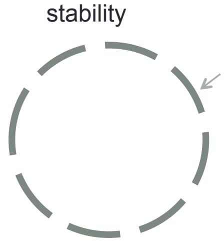
$$\Rightarrow \mathbf{M} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}$$

Sign of *f* flips in other plane

- Can build this up to describe any beam line or ring

Where we're going…

- It might seem like we would start by looking at beam lines and them move on to rings, but it turns out that there is no unique treatment of a standalone beam line
  - Depends implicitly in input beam parameters
- Therefore, we will initially solve for stable motion in a periodic system.
- The overall periodicity is usually a "ring", but that is generally divided into multiple levels of sub-periodicity, down to individual FODO cells
  - In addition to simplifying the design, we'll see that periodicity is important to stability



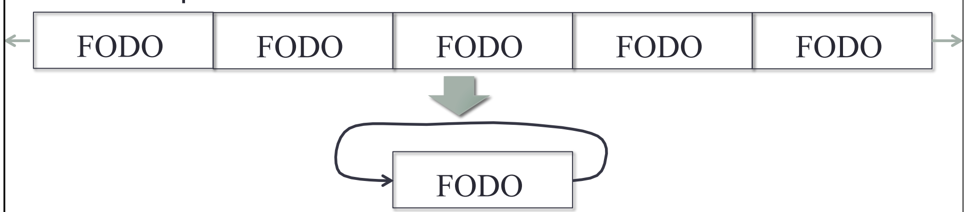
Periodic "cell"

$$\mathbf{M}_{ring} = \mathbf{M}_{cell}\mathbf{M}_{cell} \cdots \mathbf{M}_{cell} = \mathbf{M}_{cell}^N$$

- Our goal is to de-couple the problem into two parts
  - The "lattice": a mathematical description of the machine itself, based only on the magnetic fields, *which is identical for each identical cell*
  - A mathematical description of the ensemble of particles circulating in the machine ("emittance");

Periodicity

- We will build our complete system out of "cells", assuming each one to be periodic
- Example: FODO cell



- Periodicity condition $\vec{B}(x,y,s+C)=\vec{B}(x,y,s)$

"circumference" = period of cell

Quick Review of Linear Algebra

- In the absence of degeneracy, an *nxn* matrix will have *n* "eigenvectors", defined by:

$$\begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}_i = \lambda_i \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}_i$$

- Eigenvectors form an orthogonal basis
  - That is, *any* vector can be represented as a *unique sum* of eigenvectors
- In general, there exists a unitary transformation, such that

$$\mathbf{M}' = \mathbf{U}\mathbf{M}\mathbf{U}^{-1} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \rightarrow \mathbf{V}_i' = \mathbf{U}\mathbf{V}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_i$$

- Because both the trace and the determinant of a matrix are invariant under a unitary transformation:

$$\mathrm{Tr}(\mathbf{M}) = M_{11} + M_{22} + \cdots + M_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$
$$\mathrm{Det}(\mathbf{M}) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$$

Stability Criterion

- We can represent an arbitrarily complex ring as a combination of individual matrices

$$\mathbf{M}_{ring} = \mathbf{M}_n \ldots \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1$$

- We can express an arbitrary initial state as the sum of the eigenvectors of this matrix

$$\begin{pmatrix} x \\ x' \end{pmatrix} = A\mathbf{V_1} + B\mathbf{V_2} \Rightarrow \mathbf{M}\begin{pmatrix} x \\ x' \end{pmatrix} = A\lambda_1\mathbf{V}_1 + B\lambda_2\mathbf{V_2}$$

- After *n* turns, we have $\mathbf{M}^n\begin{pmatrix} x \\ x' \end{pmatrix} = A\lambda_1^n\mathbf{V_1} + B\lambda_2^n\mathbf{V_2}$

- Because the individual matrices have *unit* determinants, the product must as well, so

$$\mathrm{Det}(\mathbf{M}) = \lambda_1\lambda_2 = 1 \rightarrow \lambda_2 = 1/\lambda_1$$

Stability Criterion (cont'd)

- We can therefore express the eigenvalues as

$$\lambda_1 = e^{a}; \lambda_2 = e^{-a}; \text{ where } a \text{ is in general complex}$$

- However, if *a* has any real component, one of the solutions will grow exponentially, so the only stable values are

$$\lambda_1 = e^{i\mu}; \lambda_2 = e^{-i\mu}; \text{ where } \mu \text{ is real}$$

- Examining the (invariant) trace of the matrix

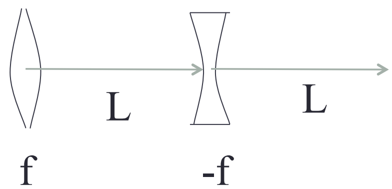
$$\mathrm{Tr}[\mathbf{M}] = e^{i\mu} + e^{-i\mu} = 2\cos\mu$$

- So the general stability criterion is simply

$$\mathrm{abs}(\mathrm{Tr}[\mathbf{M}]) < 2$$

Example

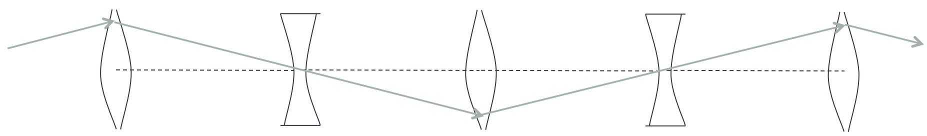
- Recall our FODO cell



$$M = \begin{pmatrix} 1-\frac{L}{f}-\left(\frac{L}{f}\right)^2 & 2L+\frac{L^2}{f} \\ -\frac{L}{f^2} & 1+\frac{L}{f} \end{pmatrix}$$

- Our stability requirement becomes

$$\text{abs}\left(2-\left(\frac{L}{f}\right)^2\right) \leq 2 \Rightarrow \boxed{L \leq 2f}$$



Twiss Parameterization

- We can express the transfer matrix for one period as the sum of an identity matrix and a traceless matrix

$$\mathbf{M}(s+C,s) = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

"Twiss Parameters" not Lorentz parameters!!

- We require the second matrix to have Det()=1, so $\boxed{-\alpha(s)^2+\beta(s)\gamma(s)=1}$
- The requirement that Det(M)=1 implies

➔ only two independent

$$A^2+B^2=1$$

- We can already identify A=Tr(M)/2=cosμ, so we can identify B=sinμ and write

$$M(s+C,s) = \cos\mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\mu\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \equiv \mathbf{I}\cos\mu + \mathbf{J}\sin\mu$$

- Note that

$$\mathbf{J}^2 = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} = \begin{pmatrix} \alpha^2(s)-\beta(s)\gamma(s) & 0 \\ 0 & \alpha^2(s)-\beta(s)\gamma(s) \end{pmatrix} = -\mathbf{I}$$

- So we can identify it with i=sqrt(-1) and write $\boxed{\mathbf{M}(s+C,s) = e^{\mu\mathbf{J}(s)}}$

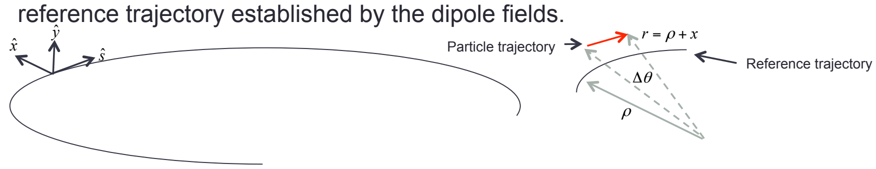
Equations of Motion

- General equation of motion

$$\vec{F} = e\vec{v}\times\vec{B} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\gamma m\dot{\vec{R}} = \gamma m\ddot{\vec{R}}$$

$$\Rightarrow \ddot{\vec{R}} = \frac{e\vec{v}\times\vec{B}}{\gamma m} = \frac{e}{\gamma m}\begin{vmatrix}\hat{x} & \hat{y} & \hat{s}\\ v_x & v_y & v_s\\ B_x & B_y & 0\end{vmatrix} = \frac{e}{\gamma m}\left(-v_s B_y\hat{x} + v_s B_x\hat{y} + \left(v_x B_y - v_y B_x\right)\hat{s}\right)$$

- For the moment, we will consider motion in the horizontal (x) plane, with a reference trajectory established by the dipole fields.



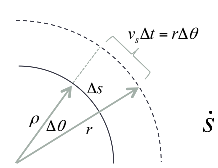
- Solving in this coordinate system, we have

$$\begin{aligned}
\vec{R} &= r\hat{x} + y\hat{y}\\
\dot{\vec{R}} &= \dot{r}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y} &&= \dot{r}\hat{x} + r\dot{\theta}\hat{s} + \dot{y}\hat{y}\\
\ddot{\vec{R}} &= \left(\ddot{r}\hat{x} + \dot{r}\dot{\hat{x}}\right) + \left(\dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}\dot{\hat{s}}\right) + \ddot{y}\hat{y} &&= \ddot{r}\hat{x} + 2\dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} - r\dot{\theta}^2\hat{x} + \ddot{y}\hat{y}\\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{x} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{s} + \ddot{y}\hat{y}
\end{aligned}$$

Equations of Motion (cont'd)

- Equating the x terms

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= -\frac{ev_sB_y}{\gamma m}\\
&= -\frac{ev_s^2B_y}{\gamma m v_s} = -\frac{ev_s^2B_y}{p}\\
&= -\frac{v_s^2B_y}{(B\rho)}
\end{aligned}$$



Note: s measured along *nominal* trajectory, v_s measured along *actual* trajectory

$$\dot{s} = \frac{ds}{dt} = \frac{\rho}{r}v_s$$

- Re-express in terms of path length s. Use

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_s\frac{\rho}{r}\frac{d}{ds} \Rightarrow \frac{d^2}{dt^2} = \left(v_s\frac{\rho}{r}\right)^2\frac{d^2}{ds^2}; \qquad \dot{\theta} = \frac{v_s}{r}$$

- Rewrite equation

$$\left(v_s\frac{\rho}{r}\right)^2\frac{d^2r}{ds^2} - r\left(\frac{v_s}{r}\right)^2 = -\frac{v_s^2B_y}{(B\rho)}$$

(rearrange terms) $\Rightarrow$

$$r'' = -\frac{v_s^2B_y}{(B\rho)}\frac{r^2}{\rho^2} + \frac{r}{\rho^2}$$

(use $r = \rho + x$) $\Rightarrow$

$$x'' = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2}; \quad y'' = \frac{B_x}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2$$

Equations of Motion

- Expand fields linearly about the nominal trajectory

$$B_x(x,y,s) = B_x(0,0,s) + \frac{\partial B_x(s)}{\partial x}x + \frac{\partial B_x(s)}{\partial y}y \xrightarrow[\text{no } x \text{ dipole}]{\text{no coupling}} \frac{\partial B_x(s)}{\partial y}y$$

$$B_y(x,y,s) = B_y(0,0,s) + \frac{\partial B_y(s)}{\partial x}x + \frac{\partial B_y(s)}{\partial y}y \xrightarrow{\text{no coupling}} B_0 + \frac{\partial B_y(s)}{\partial x}x = \frac{(B\rho)}{\rho} + \frac{\partial B_y(s)}{\partial x}x$$

- Plug into equations of motion and keep only linear terms in *x* and *y*

$$x'' = -\frac{\left(\frac{(B\rho)}{\rho} + \frac{\partial B_y(s)}{\partial x}x\right)}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2} = -\cancel{\frac{1}{\rho}} - \frac{1}{(B\rho)}\frac{\partial B_y(s)}{\partial x}x - \frac{2}{\rho^2}x + \cancel{\frac{1}{\rho}} + \frac{1}{\rho^2}x$$

$$\Rightarrow \quad x'' + \left[\frac{1}{\rho^2} + \frac{1}{(B\rho)}\frac{\partial B_y(s)}{\partial x}\right]x = 0$$

$$y'' - \frac{1}{(B\rho)}\frac{\partial B_x(s)}{\partial y}y = 0$$

Looks "kinda sorta like" a harmonic oscillator

Comment on our Equations

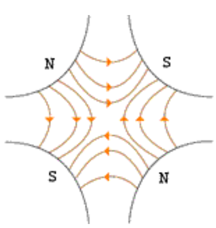
- We have our equations of motion in the form of two "Hill's Equations"

$$x'' + K_x(s)x = 0 \qquad K > 0 \Rightarrow \text{"focusing"}$$

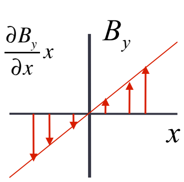
$$y'' + K_y(s)y = 0 \qquad K < 0 \Rightarrow \text{"defocusing"}$$

K(s) periodic!

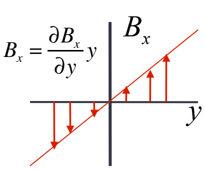
- This is the most general form for a conservative, periodic, system in which deviations from equilibrium small enough that the resulting forces are approximately linear
- In addition to the curvature term, this can only include the *linear* terms in the magnetic field (ie, the "quadrupole" term)



$$B_y = \frac{\partial B_y}{\partial x}x$$



$$B_x = \frac{\partial B_x}{\partial y}y$$



Note: $\vec{\nabla} \times \vec{B} = 0 \rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$

Comments (cont'd)

- The dipole term is *implicitly* accounted for in the definition of the reference trajectory (local curvature ρ).
- Any higher order (nonlinear) terms are dealt with as perturbations.
- Rotated quadruple ("skew") terms lead to coupling, which we won't consider yet.

General Solution

- These are second order homogeneous differential equations, so the explicit equations of motion will be linearly related to the initial conditions by

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} m_{11}(s) & m_{12}(s) \\ m_{21}(s) & m_{22}(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- Exactly as we would expect from our initial naïve treatment of the beam line elements.

Piecewise Solution

- Again, these equations are in the form $x'' + K(s)x = 0$
- For K constant, these equations are quite simple. For K>0 (focusing), it's just a harmonic oscillator and we write

$$x(s) = A\cos\left(\sqrt{K}s + \delta\right) = a\cos\left(\sqrt{K}s\right) + b\sin\left(\sqrt{K}s\right)$$
$$x'(s) = -\sqrt{K}a\sin\left(\sqrt{K}s\right) + \sqrt{K}b\cos\left(\sqrt{K}s\right)$$

- In terms if initial conditions, we identify and write $a = x_0; b = \frac{x'_0}{\sqrt{K}}$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{K}s\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}s\right) \\ -\sqrt{K}\sin\left(\sqrt{K}s\right) & \cos\left(\sqrt{K}s\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- For K<0 (defocusing), the solution becomes

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}s\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}s\right) & \cosh\left(\sqrt{|K|}s\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- For K=0 (a "drift"), the solution is simply

$$x(s) = x_0 + x'_0 s$$
$$\Rightarrow \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- We can now express the transfer matrix of an arbitrarily complex beam line with $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\mathbf{M}_3 ... \mathbf{M}_n$
- But there's a limit to what we can do with this

Closed Form Solution

- Our linear equations of motion are in the form of a "Hill's Equation"

$$x'' + K(s)x = 0; \quad K(s+C) = K(s)$$

Consider only periodic systems at the moment

- If K is a constant >0, then $x(s) = A\cos\left(\sqrt{K}s + \delta\right)$ so try a solution of the form

$$x(s) = Aw(s)\cos\left(\psi(s) + \delta\right)$$

assume $w(s+C) = w(s)$, BUT $\psi(s+C) \neq \psi(s)$

- If we plug this into the equation, we get

$$x'' + Kx = A(w'' - w\psi'^2 + Kw)\cos(\psi+\delta) - A(2w'\psi' + w\psi'')\sin(\psi+\delta) = 0$$

- Coefficients must independently vanish, so the sin term gives

$$2w'\psi' + w\psi'' = 0 \xrightarrow{\text{mutliply by } w} 2ww'\psi' + w^2\psi'' = \left(w^2\psi'\right)' = 0 \Rightarrow \psi' = \frac{k}{w^2}$$

- If we re-express our general solution

$$x = w\left(A_1\cos\psi + A_2\sin\psi\right)$$
$$x' = \left(A_1 w' + A_2 w\psi'\right)\cos\psi + \left(A_2 w' - A_1 w\psi'\right)\sin\psi$$
$$= \left(A_1 w' + A_2\frac{k}{w}\right)\cos\psi + \left(A_2 w' - A_1\frac{k}{w}\right)\sin\psi$$

$$w'' - \frac{k^2}{w^3} + Kw = 0$$

We'll see this much later

Solving for periodic motion

- Plug in initial condition ($s=0 \rightarrow \Psi=0$)
- Define phase advances over one period

$$S_C \equiv \sin\psi(C); C_C \equiv \cos\psi(C)$$

and we have

$$A_1 = \frac{x_0}{w}$$
$$A_2 = \frac{x_0' w - x_0 w'}{k}$$

$$x(C) = x_0 C_C + \left(\frac{x_0' w^2 - x_0 ww'}{k}\right)S_C$$
$$= \left(C_C - \frac{w'w}{k}S_C\right)x_0 + \frac{w^2}{k}S_C x_0'$$
$$x'(C) = x_0' C_C + \left(\frac{x_0' ww'}{k} - \frac{x_0 w'^2}{k} - \frac{x_0 k}{w^2}\right)S_C$$
$$= \left(-\frac{k}{w^2} - \frac{w'^2}{k}\right)S_C x_0 + \left(C_C + \frac{ww'}{k}S_C\right)x'$$

$$\Rightarrow \begin{pmatrix} x(C) \\ x'(C) \end{pmatrix} = \begin{pmatrix} \left(C_C - \frac{w'w}{k}S_C\right) & \frac{w^2}{k}S_C \\ \left(-\frac{1 + \left(\frac{ww'}{k}\right)^2}{w^2/k}\right)S_C & \left(C_C + \frac{ww'}{k}S_C\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

This form will make sense in a minute

- But wait! We've seen this before…

Twiss representation of a Period

- We showed a flew slides ago, that we could write

$$\mathbf{M}(s+C,s) = \cos\mu\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin\mu\begin{pmatrix}\alpha(s) & \beta(s)\\ -\gamma(s) & -\alpha(s)\end{pmatrix}$$
$$= \begin{pmatrix}\cos\mu+\alpha(s)\sin\mu & \beta(s)\sin\mu\\ -\gamma(s)\sin\mu & \cos\mu-\alpha(s)\sin\mu\end{pmatrix}$$

- We quickly identify

$$\mu = \psi(C)$$

Phase advance over one period

$$\beta(s) = \frac{w^2}{k}$$
$$\alpha(s) = -\frac{ww'}{k} = -\frac{1}{2}\frac{d}{ds}\left(\frac{w^2}{k}\right) = -\frac{1}{2}\frac{d\beta(s)}{ds}$$
$$\gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

β and γ always positive!

- We also showed some time ago that a requirement of the Hill's Equation was that

$$\psi' = \frac{k}{w^2} = \frac{1}{\beta(s)} \Rightarrow \psi(C) = \mu = \int_{s_0}^{s_0+C}\frac{1}{\beta(s)}ds$$

Super important! Remember forever!

Solving for the Lattice Functions

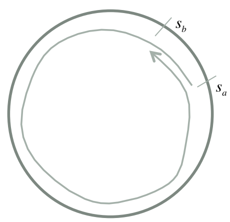
- If we calculate (or measure) the transfer matrix of a period, we can solve for the lattice functions at the ends

$$\mathbf{M} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix} = \begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}$$
$$\cos\mu = \frac{1}{2}Tr(\mathbf{M}) = \frac{1}{2}(M_{11}+M_{22})$$
$$\sin\mu = \mathrm{sign}(M_{12})\sqrt{1-\cos^2\mu}$$
$$\beta = \frac{M_{12}}{\sin\mu}$$
$$\alpha = \frac{M_{11}-\cos\mu}{\sin\mu}$$
$$\gamma = \frac{1+\alpha^2}{\beta}$$

Sign of M_{12} resolves sign ambiguity of sine function

Evolution of the Lattice Functions

- If we know the lattice functions at one point, we can use the transfer matrix to transfer them to another point by considering the following two equivalent things
  - Going around the period, starting and ending at point *a*, then proceeding to point *b*
  - Going from point *a* to point *b*, *then* going all the way around the period



$$\mathbf{M}(s_b+C,s_b)\mathbf{M}(s_b,s_a)=\mathbf{M}(s_b,s_a)\mathbf{M}(s_a+C,s_a)$$
$$\mathbf{M}(s_b+C,s_b)=\mathbf{M}(s_b,s_a)\mathbf{M}(s_a+C,s_a)\mathbf{M}^{-1}(s_b,s_a)$$

Recall:
$$\mathbf{J}(s)\equiv\begin{pmatrix}\alpha(s) & \beta(s)\\ -\gamma(s) & -\alpha(s)\end{pmatrix}$$

$$\mathbf{M}(s+C,s)=\mathbf{I}\cos\mu+\mathbf{J}(s)\sin\mu$$
$$\mathbf{I}\cos\mu+\mathbf{J}(s_b)\sin\mu=\mathbf{M}(s_b,s_a)\left(\mathbf{I}\cos\mu+\mathbf{J}(s_a)\sin\mu\right)\mathbf{M}^{-1}(s_b,s_a)$$
$$=\mathbf{I}\cos\mu+\left(\mathbf{M}(s_b,s_a)\mathbf{J}(s_a)\mathbf{M}^{-1}(s_b,s_a)\right)\sin\mu$$
$$\equiv\mathbf{I}\cos\mu+\mathbf{J}(s_b)\sin\mu$$
$$\Rightarrow\mathbf{J}(s_b)=\mathbf{M}(s_b,s_a)\mathbf{J}(s_a)\mathbf{M}^{-1}(s_b,s_a)$$

Evolution of the Lattice functions (cont'd)

- Using $\mathbf{M}(s_b,s_a)=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\Rightarrow\mathbf{M}^{-1}(s_b,s_a)\begin{pmatrix}m_{22} & -m_{12}\\ -m_{21} & m_{11}\end{pmatrix}$
- We can now evolve the **J** matrix at any point as

$$\mathbf{J}(s_b)=\begin{pmatrix}\alpha(s_b) & \beta(s_b)\\ -\gamma(s_b) & -\alpha(s_b)\end{pmatrix}=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}\alpha(s_a) & \beta(s_a)\\ -\gamma(s_a) & -\alpha(s_a)\end{pmatrix}\begin{pmatrix}m_{22} & -m_{12}\\ -m_{21} & m_{11}\end{pmatrix}$$

- Multiplying this mess out and gathering terms, we get

$$\begin{pmatrix}\alpha(s_b)\\ \beta(s_b)\\ \gamma(s_b)\end{pmatrix}=\begin{pmatrix}(m_{11}m_{22}+m_{12}m_{21}) & (-m_{11}m_{21}) & (-m_{12}m_{22})\\ (-2m_{11}m_{12}) & (m_{11}^2) & (m_{12}^2)\\ (-2m_{21}m_{22}) & (m_{21}^2) & (m_{22}^2)\end{pmatrix}\begin{pmatrix}\alpha(s_a)\\ \beta(s_a)\\ \gamma(s_a)\end{pmatrix}$$

Examples

- Drift of length L:

$$\mathbf{M}=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha(s)\\ \beta(s)\\ \gamma(s)\end{pmatrix}=\begin{pmatrix}1 & 0 & -s\\ -2s & 1 & s^2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha(0)\\ \beta(0)\\ \gamma(0)\end{pmatrix}\Rightarrow\begin{matrix}\alpha(s)=\alpha_0-\gamma_0 s\\ \beta(s)=\beta_0-2\alpha_0 s+\gamma_0 s^2\\ \gamma(s)=\gamma_0\end{matrix}$$

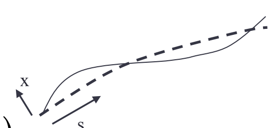
- Thin focusing (defocusing) lens:

$$\mathbf{M}=\begin{pmatrix}1 & 0\\ \mp\frac{1}{f} & 1\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha'\\ \beta'\\ \gamma'\end{pmatrix}=\begin{pmatrix}1 & \pm\frac{1}{f} & 0\\ 0 & 1 & 0\\ \pm\frac{2}{f} & \frac{1}{f^2} & 1\end{pmatrix}\begin{pmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{pmatrix}\Rightarrow\begin{matrix}\alpha'=\alpha_0\pm\frac{1}{f}\beta_0\\ \beta'=\beta_0\\ \gamma'=\gamma_0\pm\frac{2}{f}\alpha_0+\frac{1}{f^2}\beta_0\end{matrix}$$

Betatron motion

- Generally, we find that we can describe particle motion in terms of initial conditions and a "beta function" $\beta(s)$, which is only a function of location in the nominal path.

$$\beta(s)\propto w(s)^2$$



Lateral deviation in one plane

$$x(s)=A\sqrt{\beta(s)}\cos\left(\psi(s)+\delta\right)$$

Phase advance

$$\psi(s)=\int_0^s\frac{ds}{\beta(s)}$$

The "betatron function" $\beta(s)$ is effectively the local wavenumber and also defines the beam envelope.

Closely spaced strong quads -> small β -> small aperture, lots of wiggles

Sparsely spaced weak quads -> large β -> large aperture, few wiggles

Behavior Over Multiple Turns

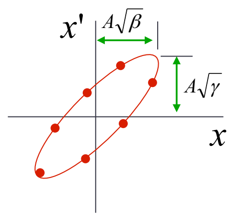
- The general expressions for motion are $x = A\sqrt{\beta}\cos\phi; \phi = \psi(s) + \delta$

$$x' = -\frac{A}{\sqrt{\beta}}(\alpha\cos\phi + \sin\phi)$$

- We form the combination

$$\gamma x^2 + 2\alpha x x' + \beta x'^2$$
$$= A^2\left(\gamma\ \beta\cos^2\phi - 2\alpha^2\cos^2\phi - 2\alpha\sin\phi\cos\phi + \alpha^2\cos^2\phi + \sin^2\phi + 2\alpha\sin\phi\cos\phi\right)$$
$$= A^2\left(\left(\gamma\ \beta - \alpha^2\right)\cos^2\phi + \sin^2\phi\right)$$
$$= A^2 = \text{constant}$$

- This is the equation of an ellipse.

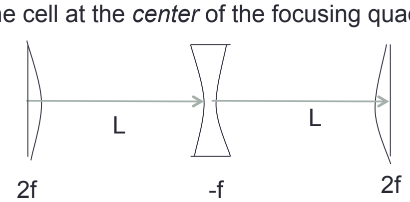


Area = πA^2

Particle will return to a *different* point on the *same* ellipse each time around the ring.

Symmetric Treatment of FODO Cell

- If we evaluate the cell at the *center* of the focusing quad, it looks like



Leading to the transfer Matrix

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L\left(1 + \frac{L}{2f}\right) \\ -\frac{L}{2f^2} + \frac{L^2}{4f^3} & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

Note: some textbooks have *L*=total length

Lattice Functions in FODO Cell

We know from our Twiss Parameterization that this can be written as

$$\begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L\left(1+\frac{L}{2f}\right) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} \end{pmatrix} = \begin{pmatrix} \cos\mu+\alpha_0\sin\mu & \beta_0\sin\mu \\ -\gamma_0\sin\mu & \cos\mu-\alpha_0\sin\mu \end{pmatrix}$$

From which we see that the Twiss functions *at the middle of the magnets* are

$$\alpha_0 = 0$$

$$\cos\mu = 1-\frac{L^2}{2f^2} = 1-2\sin^2\frac{\mu}{2}$$

$$\rightarrow \sin\frac{\mu}{2} = \frac{L}{2f}$$

$$\beta_0\sin\mu = 2L\left(1+\frac{L}{2f}\right)$$

$$\rightarrow \beta_0 = 2L\frac{\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu} = \beta_{max}$$

recall

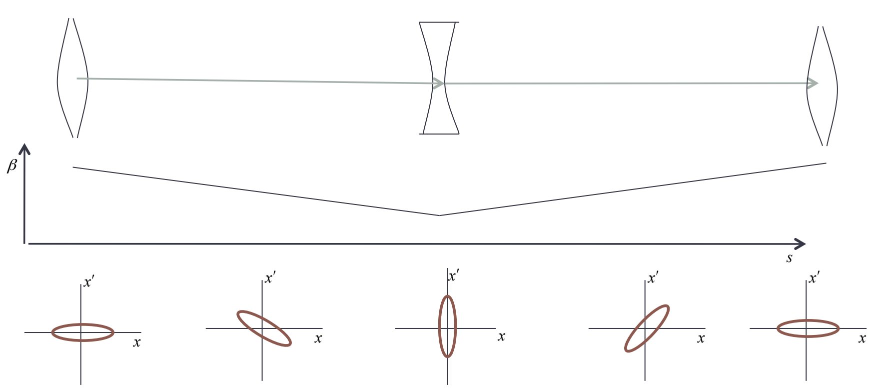
$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} \qquad \gamma\beta-\alpha^2 = 1 \rightarrow \gamma_0 = \frac{1}{\beta_0}$$

Flip sign of f to get other plane

$$\beta_{min} = 2L\frac{\left(1-\sin\frac{\mu}{2}\right)}{\sin\mu}$$

Lattice Function in FODO Cell (cont'd)

- As particles go through the lattice, the Twiss parameters will vary periodically:



β = max	β = decreasing	β = min	β = increasing	
α = 0	α >0	α = 0	α < 0	
➔maximum	➔focusing	➔minimum	➔defocusing	

Motion at each point bounded by $|x(s)| \leq A\sqrt{\beta(s)}$

Interlude: Some Formalism

- Let's look at the Hill's equation again… $x''+K(s)x=0$
- We can write the general solution as a linear combination of a "sine-like" and "cosine-like" term $x(s)=aC(s)+bS(s)$ where

$$C(0)=1;S(0)=0$$
$$C'(0)=0;S'(0)=1$$

- When we plug this into the original equation, we see that

$$a\big(C''(s)+K(s)C(s)\big)+b\big(S''(s)+K(s)S(s)\big)=0$$

- Since *a* and *b* are arbitrary, each function must *independently* satisfy the equation. We further see that when we look at our initial conditions

$$x(0)=aC(0)+bS(0)=a\Rightarrow a=x_0$$
$$x'(0)=aC'(0)+bS'(0)=b\Rightarrow b=x'_0$$

- So our transfer matrix becomes

$$\begin{pmatrix}x(s)\\x'(s)\end{pmatrix}=\mathbf{M}\begin{pmatrix}x_0\\x'_0\end{pmatrix}\Rightarrow\begin{pmatrix}x(s)\\x'(s)\end{pmatrix}=\begin{pmatrix}C(s)&S(s)\\C'(s)&S'(s)\end{pmatrix}\begin{pmatrix}x_0\\x'_0\end{pmatrix}$$

Closing the Loop

- We've got a general equation of motion in terms of initial conditions and a single "betatron function" β(s)

$$x(s)=A\sqrt{\beta(s)}\cos\big(\psi(s)+\delta\big);\quad \psi(s)=\int_0^s\frac{ds}{\beta(s)}$$

- Important note!
  - *β(s)* (and therefore *α(s) and γ(s)*) are defined to have the periodicity of the machine!
  - In general *Ψ(s)* (and therefore *x(s)*) DO NOT!
    - Indeed, we'll see it's very bad if they do

Define "tune" as the number of pseudo-oscillations around the ring

$$\nu\equiv\frac{1}{2\pi}\oint\frac{ds}{\beta(s)}=\frac{1}{2\pi}N_{cell}\mu_{cell}$$

- So far, we have used the lattice functions at a point *s* to propagate the particle to the *same point* in the next period of the machine. We now generalize this to transport the beam from one point to another, knowing only initial conditions and the lattice functions at both points

$$x(s)=A\sqrt{\beta(s)}\cos\big(\psi(s)+\delta\big)$$
$$x'(s)=A\frac{1}{2}\frac{1}{\sqrt{\beta(s)}}\frac{d\beta(s)}{ds}\cos\big(\psi(s)+\delta\big)-A\sqrt{\beta(s)}\frac{d\psi(s)}{ds}\sin\big(\psi(s)+\delta\big)$$
$$=-A\frac{1}{\sqrt{\beta(s)}}\Big(\alpha(s)\cos\big(\psi(s)+\delta\big)+\sin\big(\psi(s)+\delta\big)\Big)$$

- We use this to define the trigonometric terms at the initial point as

$$C_0 \equiv \cos\left(\psi(s_0)+\delta\right) = x_0 \frac{1}{A\sqrt{\beta_0}}$$

$$S_0 \equiv \sin\left(\psi(s_0)+\delta\right) = -x_0' \frac{\sqrt{\beta_0}}{A} - x_0 \frac{\alpha_0}{A\sqrt{\beta_0}}$$

- We can then use the sum angle formulas to define the trigonometric terms at any point $\Psi(s_1)$ as

$$\begin{aligned}
\cos\left(\psi(s_1)+\delta\right) \equiv C_1 &= \cos\left(\psi(s_0)+\delta+\Delta\psi\right) = C_0\cos\Delta\psi - S_0\sin\Delta\psi \\
&= x_0\left(\frac{1}{A\sqrt{\beta_0}}\cos\Delta\psi + \frac{\alpha_0}{A\sqrt{\beta_0}}\sin\Delta\psi\right) + x_0'\frac{\sqrt{\beta_0}}{A}\sin\Delta\psi \\
\sin\left(\psi(s_1)+\delta\right) \equiv S_1 &= \sin\left(\psi(s_0)+\delta+\Delta\psi\right) = S_0\cos\Delta\psi + C_0\sin\Delta\psi \\
&= x_0\left(\frac{1}{A\sqrt{\beta_0}}\sin\Delta\psi - \frac{\alpha_0}{A\sqrt{\beta_0}}\cos\Delta\psi\right) - x_0'\frac{\sqrt{\beta_0}}{A}\sin\Delta\psi
\end{aligned}$$

General Transfer Matrix

- We plug the previous angular identities for C_1 and S_1 into the general transport equations

$$x_1 = A\sqrt{\beta_1}\cos\left(\psi_1+\delta\right) = A\sqrt{\beta_1}C_1$$

$$x_1' = -A\frac{1}{\sqrt{\beta_1}}\left(\alpha_1 C_1 + S_1\right)$$

And (after a little tedious algebra) we find

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\left(\cos\Delta\psi + \alpha_0\sin\Delta\psi\right) & \sqrt{\beta_0\beta_1}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta_1}}\left(\left(\alpha_0-\alpha_1\right)\cos\Delta\psi - \left(1+\alpha_0\alpha_1\right)\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta_1}}\left(\cos\Delta\psi - \alpha_1\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- This is a mess, but we'll often restrict ourselves to the extrema of β, where

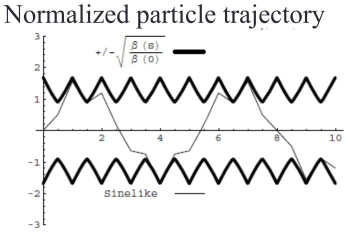
$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\cos\Delta\psi & \sqrt{\beta_0\beta_1}\sin\Delta\psi \\ -\frac{1}{\sqrt{\beta_0\beta_1}}\sin\Delta\psi & \sqrt{\frac{\beta_0}{\beta_1}}\cos\Delta\psi \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

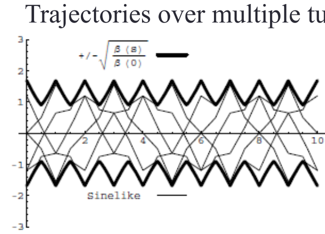
Conceptual understanding of β

- It's important to remember that the betatron function represents a *bounding envelope* to the beam motion, not the beam motion itself

Normalized particle trajectory



Trajectories over multiple turns



$$x(s) = A[\beta(s)]^{1/2}\sin(\psi(s)+\delta)$$

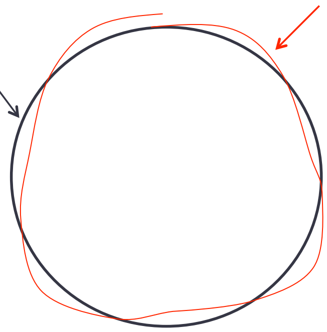
$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$\beta(s)$ is also effectively the local wave number which determines the rate of phase advance

Closely spaced strong quads ➔ small β ➔ small aperture, lots of wiggles

Sparsely spaced weak quads ➔ large β ➔ large aperture, few wiggles

Betatron Tune



- As particles go around a ring, they will undergo a number of betatrons oscillations *ν* (sometimes *Q*) given by

$$\nu = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

- This is referred to as the "tune"

- We can generally think of the tune in two parts:

Integer : magnet/ aperture optimization ↗ 6.7 ↖ Fraction: Beam Stability

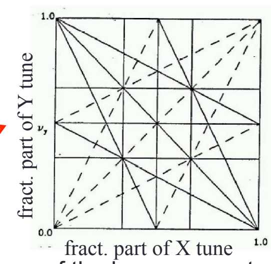
Tune, Stability, and the Tune Plane

- If the tune is an integer, or low order rational number, then the effect of any imperfection or perturbation will tend be reinforced on subsequent orbits.
- When we add the effects of coupling between the planes, we find this is also true for *combinations* of the tunes from both planes, so in general, we want to avoid

$$k_x \nu_x \pm k_y \nu_y = \text{integer} \Rightarrow \text{(resonant instability)}$$

"small" integers

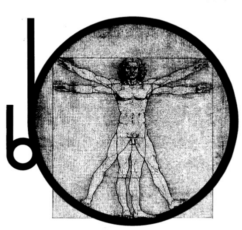
➔Avoid lines in the "tune plane"



- Many instabilities occur when something perturbs the tune of the beam, or part of the beam, until it falls onto a resonance, thus you will often hear effects characterized by the "tune shift" they produce.

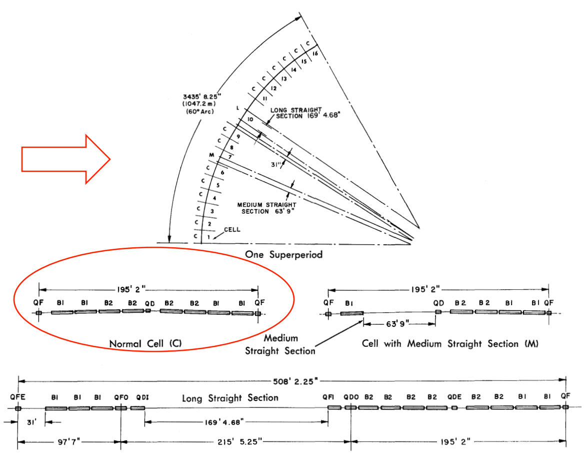
Example: Fermilab Main Ring

- First "separated function" lattice
- 1 km in radius
- First accelerated protons from 8 to 400 GeV in 1972



design report
national
accelerator
laboratory

1968



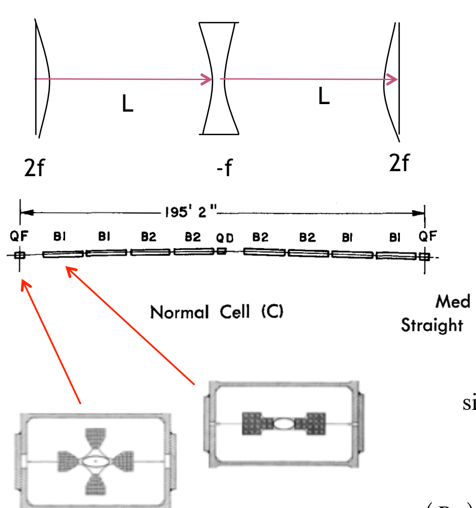
Beam Parameters

- The Main Ring accelerated protons from kinetic energy of 8 to 400 GeV*

Parameter	Symbol	Equation	Injection	Extraction
proton mass	m [GeV/c²]		0.93827	
kinetic energy	K [GeV]		8	400
total energy	E [GeV]	$K+mc^2$	8.93827	400.93827
momentum	p [GeV/c]	$\sqrt{E^2-(mc^2)^2}$	8.88888	400.93717
rel. beta	β	$(pc)/E$	0.994475	0.999997
rel. gamma	γ	$E/(mc^2)$	9.5263	427.3156
beta-gamma	βγ	$(pc)/(mc^2)$	9.4736	427.3144
rigidity	(Bρ) [T-m]	p[GeV]/(.2997)	29.65	1337.39

*remember this for problem set

Cell Parameters



- From design report
  - L=29.74 m
  - Phase advance μ=71°
  - Quad Length l_{quad}=2.13 m
- Beta functions (slide 36)

$$\beta_{\max,\min} = 2L\frac{\left(1\pm\sin\frac{\mu}{2}\right)}{\sin\mu}$$

$$= 2(29.74)\frac{\left(1\pm\sin\frac{71^\circ}{2}\right)}{\sin 71^\circ}$$

$$\boxed{= 99.4 \text{ m},\ 26.4 \text{ m}}$$

- Magnet focal length

$$\sin\frac{\mu}{2} = \frac{L}{2f} \rightarrow f = \frac{29.74}{2\sin(71^\circ/2)} = \boxed{25.61 \text{ m}}$$

- Quad gradient (slide 12)

$$f = \frac{(B\rho)}{l_{quad}B'} \rightarrow B' = \frac{(B\rho)}{l_{quad}f} = \frac{(1337.39)}{(2.13)(25.6)} = \boxed{24.5 \text{ T/m}}$$

Beam Line Calculation: MAD

- There have been and continue to be countless accelerator modeling programs; however MAD ("Methodical Accelerator Design"), started in 1990, continues to be the "Lingua Franca"

main_ring.madx

```
!
! One FODO cell from the FNAL Main Ring (NAL Design Report, 1968)
!
beam, particle=proton,energy=400.938272,npart=1.0E9;

LQ:=1.067;
LD:=29.74-2*LQ;

qf: QUADRUPOLE, L=LQ, K1=.0195;
d: DRIFT, L=LD;
qd: QUADRUPOLE, L=LQ, K1=-.0195;

fodo: line = (qf,d,qd,qd,d,qf);
use, period=fodo;

match,sequence=FODO;

SELECT,FLAG=SECTORMAP,clear;
SELECT,FLAG=TWISS,column=name,s,betx,alfx,bety,alfy,mux,muy;
TWISS,SAVE;

PLOT,interpolate=true,,colour=100,HAXIS=S, VAXIS1=BETX,BETY;
PLOT,interpolate=true,,colour=100,HAXIS=S, VAXIS1=ALFX,ALFY;

stop;
```

half quad K1=1/(2f)/L

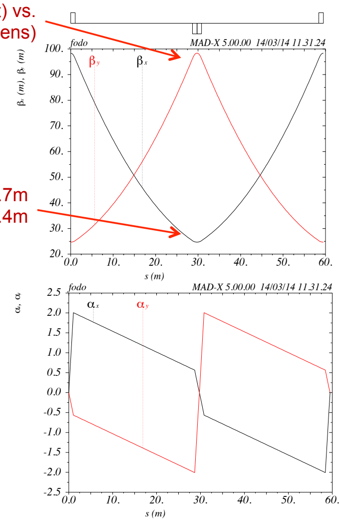
build FODO cell

force periodicity

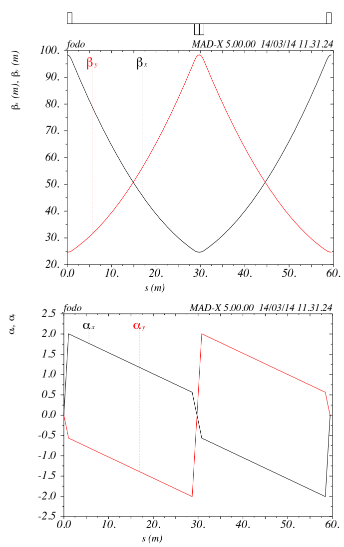
calculate Twiss parameters

98.4m (exact) vs. 99.4m (thin lens)

24.7m vs. 26.4m



Evolution if Twiss Parameters (slide 31)



- Thin focusing (defocusing) lens

$$\mathbf{M}=\begin{pmatrix}1 & 0\\ \mp\frac{1}{f} & 1\end{pmatrix} \Rightarrow \begin{aligned}\alpha' &= \alpha_0 \pm \frac{1}{f}\beta_0\\ \beta' &= \beta_0\\ \gamma' &= \gamma_0 \pm \frac{2}{f}\alpha_0 + \frac{1}{f^2}\beta_0\end{aligned}$$

- Drift

$$\mathbf{M}=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix} \Rightarrow \begin{aligned}\alpha(s) &= \alpha_0 - \gamma_0 s\\ \beta(s) &= \beta_0 - 2\alpha_0 s + \gamma_0 s^2\\ \gamma(s) &= \gamma_0\end{aligned}$$

Digression: Intuitive vs. Counterintuitive

- We have derived the generic transfer matrix to be

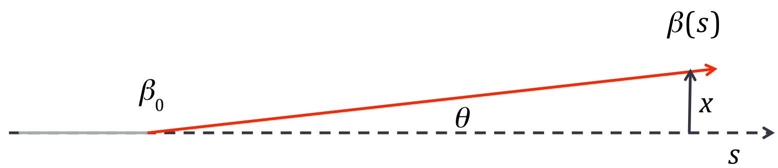
$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_0\beta_1}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta_1}}\left((\alpha_0 - \alpha_1)\cos\Delta\psi - (1+\alpha_0\alpha_1)\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta_1}}\left(\cos\Delta\psi - \alpha_1\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- Which means a particle on the axis ($x_0=0$) with an angle $x'=\theta$ will follow a path given by

$$x = \sqrt{\beta_0\beta(s)}\sin\Delta\psi(s)\theta$$

- But a particle doesn't "know" about lattice functions, so does this make sense

Example: Initial angle in a Drift Section



- Before you knew anything about beam physics, you would have said

$$x = s\theta$$

- Now you would say

$$x = \sqrt{\beta_0\beta(s)}\sin\Delta\psi(s)\theta$$

- How can both of these be true?

Evolution of Phase Angle

$$\alpha(s)=\alpha_0-\gamma_0 s$$
$$\beta(s)=\beta_0-2\alpha_0 s+\gamma_0 s^2$$
$$\gamma(s)=\gamma_0$$

$$\Delta\psi=\int_0^s\frac{ds}{\beta(s)}=\int_0^s\frac{ds}{\gamma_0 s^2-2\alpha_0 s+\beta_0}=\frac{1}{\gamma_0}\int_0^s\frac{ds}{s^2-2\frac{\alpha_0}{\gamma_0}s+\frac{\beta_0}{\gamma_0}}$$

$$=\frac{1}{\gamma_0}\int_0^s\frac{ds}{\left(s-\frac{\alpha_0}{\gamma_0}\right)^2-\frac{\alpha_0^2}{\gamma^2}+\frac{\beta_0}{\gamma_0}}=\frac{1}{\gamma_0}\int_0^s\frac{ds}{\left(s-\frac{\alpha_0}{\gamma_0}\right)^2+\frac{-\alpha_0^2+\gamma_0\beta_0}{\gamma_0^2}}=\frac{1}{\gamma_0}\int_0^s\frac{ds}{\left(s-\frac{\alpha_0}{\gamma_0}\right)^2+\frac{1_0}{\gamma_0^2}}$$

$$=\tan^{-1}\left(\gamma_0 s-\alpha_0\right)\Big|_0^s=\tan^{-1}\left(\gamma_0 s-\alpha_0\right)-\tan^{-1}\left(-\alpha_0\right)$$

$$=\tan^{-1}\left(-\alpha_0(s)\right)-\tan^{-1}\left(-\alpha_0\right)$$

$$\equiv\psi(s)-\psi_0$$

$$\int\frac{dx}{x^2+u^2}=\frac{1}{u}\tan^{-1}\left(\frac{x}{u}\right)$$

$$\alpha(s)\to-\infty\Rightarrow\psi(s)\to\frac{\pi}{4}$$

Sine of Phase Angle

$$\sin\theta=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$
$$\cos\theta=\frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\sin\Delta\psi=\sin\left(\psi(s)-\psi_0\right)=\sin\psi(s)\cos\psi_0-\cos\psi(s)\sin\psi_0$$

$$=\frac{\tan\psi(s)}{\sqrt{1+\tan^2\psi(s)}\sqrt{1+\tan^2\psi_0}}-\frac{\tan\psi_0}{\sqrt{1+\tan^2\psi(s)}\sqrt{1+\tan^2\psi_0}}=\frac{\tan\psi(s)-\tan\psi_0}{\sqrt{1+\tan^2\psi(s)}\sqrt{1+\tan^2\psi_0}}$$

$$=\frac{-\alpha(s)+\alpha_0}{\sqrt{1+\alpha^2(s)}\sqrt{1+\alpha_0^2}}=\frac{-\alpha_0+\gamma_0 s+\alpha_0}{\sqrt{\gamma(s)\beta(s)}\sqrt{\gamma_0\beta_0}}=\frac{\gamma_0 s}{\sqrt{\gamma_0\beta(s)}\sqrt{\gamma_0\beta_0}}$$

$$=\frac{s}{\sqrt{\beta(s)\beta_0}}$$

$$\gamma(s)=\gamma_0$$

$$\longrightarrow\quad x=\sqrt{\beta_0\beta(s)}\frac{s}{\sqrt{\beta(s)\beta_0}}\theta$$

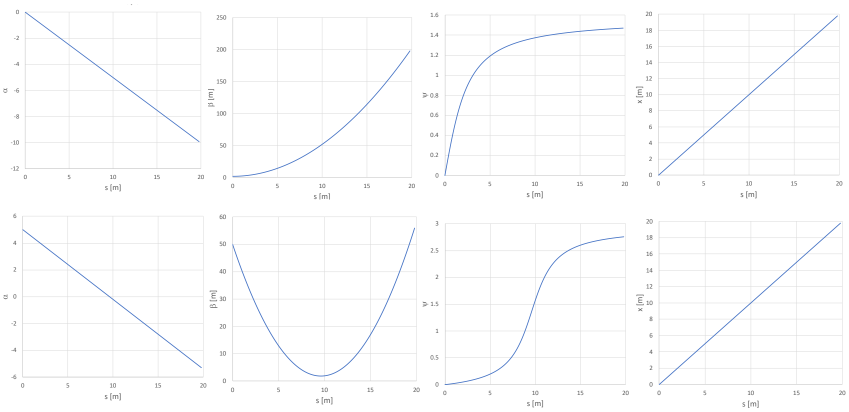
$$=s\theta$$

Examples: Different Initial Conditions

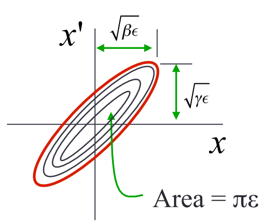
$\alpha(s)$ $\beta(s)$ $\Delta\psi(s)$ $\sqrt{\beta_0\beta(s)}\sin\Delta\psi(s)$

$\beta_0 = 2\text{m}$
$\alpha_0 = 0$

$\beta_0 = 50\text{m}$
$\alpha_0 = 5$



Characterizing an Ensemble: Emittance



If each particle is described by an ellipse with a particular amplitude, then an *ensemble* of particles will always remain within a bounding ellipse of a particular area:

$$\gamma x^2 + 2\alpha x x' + \beta x' = \epsilon \text{ or } \frac{\epsilon}{\pi}$$

Either leave the π out, or include it explicitly as a “unit”. Thus

- microns (CERN) and
- π-mm-mr (FNAL)

These are really the same

Are actually the same units (just remember you’ll *never have to explicity use π in the calculation*)

Definitions of Emittance

- Because distributions normally have long tails, we have to adopt a convention for defining the emittance. The two most common are
  - Gaussian (electron machines, CERN):

$$\epsilon = \frac{\sigma_x^2}{\beta_x}; \text{ contains 39\% of the beam}$$

$$\rightarrow \sigma_x = \sqrt{\beta_x \epsilon}$$

  - 95% Emittance (FNAL):

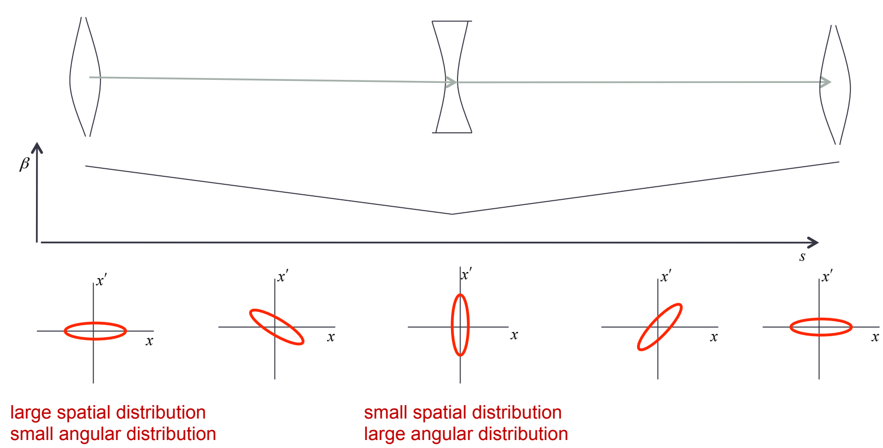
$$\epsilon_{95} = \frac{6\sigma_x^2}{\beta_x}; \text{ contains 95\% of the beam}$$

$$\rightarrow \sigma_x = \sqrt{\frac{\beta_x \epsilon_{95}}{6}}$$

- In general, emittance can be different in the two planes, but we won't worry about that for now.

Emittance and Beam Distributions

- As we go through a lattice, the bounding emittance remains constant



Distributions, Emittance, and Twiss Parameters

- The relationship between the lattice functions, RMS emittance and moment distributions is

$$\sigma_x = \sqrt{\frac{1}{N}\sum x^2} = \sqrt{\beta\epsilon}$$

$$\sigma_{x'} = \sqrt{\frac{1}{N}\sum x'^2} = \sqrt{\gamma\epsilon}$$

$$\sigma_{xx'} = \frac{1}{N}\sum xx' = -\alpha\epsilon$$

- We can turn this around to calculate the emittance and lattice functions based on measured distributions

$$\sigma_x^{\ 2}\sigma_{x'}^{\ 2} - \sigma_{xx'}^{\ 2} = \left(\beta\gamma - \alpha^2\right)\epsilon^2 = \epsilon^2 \Rightarrow$$

$$\epsilon = \sqrt{\sigma_x^{\ 2}\sigma_{x'}^{\ 2} - \sigma_{xx'}^{\ 2}}$$

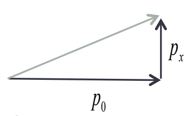
$$\beta = \frac{\sigma_x^2}{\epsilon}$$

$$\gamma = \frac{\sigma_{x'}^2}{\epsilon}$$

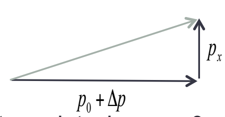
$$\alpha = -\frac{\sigma_{xx'}}{\epsilon}$$

Adiabatic Damping*

- In our discussions up to now, we assume that all fields scale with momentum, so our lattice remains the same, but what happens to the ensemble of particles? Consider what happens to the slope of a particle as the forward momentum incrementally increases.



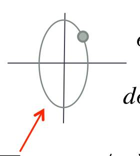
$$x' = \frac{p_x}{p_0}$$



$$x' + \Delta x' = \frac{p_x}{p_0 + \Delta p} \approx x'\left(1 - \frac{\Delta p}{p_0}\right)$$

$$\Rightarrow \Delta x' = -x'\frac{\Delta p}{p_0}$$

- If we evaluate the emittance at a point where α=0, we have



$$x' = \sqrt{\epsilon\gamma_T}\sin(\psi+\delta) = \sqrt{\frac{\epsilon}{\beta_T}}\sin(\psi+\delta)$$

$$\epsilon = \left(\gamma_T x^2 + \beta_T x'^2\right)$$

$$d\epsilon = 2\beta_T x' dx' = -2\beta_T x'^2\frac{dp}{p} = -2\epsilon\frac{dp}{p}\sin^2\left(\psi+\delta\right)$$

$$\langle d\epsilon\rangle = -2\epsilon\frac{dp}{p}\langle\sin^2\left(\psi+\delta\right)\rangle = -\epsilon\frac{dp}{p} \Rightarrow p\,d\epsilon + \epsilon\,dp = 0 \Rightarrow \epsilon p = \text{constant} = \epsilon_0 p_0$$

$$\epsilon = \epsilon_0\frac{p_0}{p} = \frac{\left(\epsilon_0\gamma_0\beta_0\right)}{\gamma\beta} \equiv \boxed{\frac{\epsilon_N}{\gamma\beta}}$$

"Normalized emittance" =constant!

*only true for protons!

Consequences of Adiabatic Damping

- As a beam is accelerated, the *normalized emittance* remains constant
  - Actual emittance goes down down $\epsilon = \frac{\epsilon_N}{\beta\gamma} \propto \frac{1}{p}$
  - Which means the actual beam size goes down as well

betatron function — RMS emittance — 95% emittance

$$\sigma_x = \sqrt{\beta_x \epsilon} = \sqrt{\frac{\beta_x \epsilon_N}{\beta\gamma}} \text{ or } \sqrt{\frac{\beta_x \epsilon_{95}}{6\beta\gamma}} \propto \frac{1}{\sqrt{p}}$$

v/c

- The angular distribution at an extremum (α=0) is

$$\sigma_{x'} = \sqrt{\frac{\gamma_x \epsilon_N}{\beta\gamma}} = \sqrt{\frac{\epsilon_N}{\beta_x \beta\gamma}}$$

- We *almost* always use normalized emittance for protons

Example: FNAL Main Ring Revisited

- We normally use 95% emittance at Fermilab, and 95% normalized emittance of the beam going into the Main Ring was about 12 π-mm-mr, so the normalized RMS emittance would be

$$\epsilon_{RMS} \approx \frac{\epsilon_{95}}{6} = 2\ \pi\text{-mm-mrad} = 2\times 10^{-6}\ \text{m}$$ ← We have divided out the "π"

- We combine this with the equations (slide 45), beam parameters (slide 47) and lattice functions (slide 48) to calculate the beam sizes at injection and extraction.

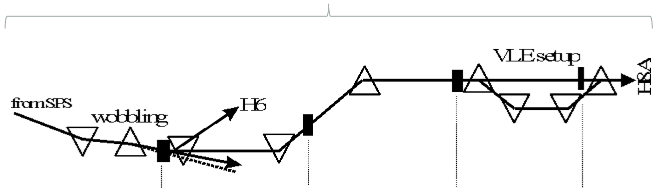
Parameter	Symbol	Equation	Injection	Extraction
kinetic energy	K [GeV]		8	400
beta-gamma	βγ		9.4736	427.3144
normalized emittance	ϵ_N [m]		$2\text{x}10^{-6}$	
beta at QF	$\beta_{\max}$ [m]		99.4	
beta at QD	$\beta_{\min}$ [m]		26.4	
x size at QF	σ_x [mm]	$\sqrt{\frac{\beta_{\max}\epsilon_N}{\beta\gamma}}$	4.58	.68
y size at QF	σ_y [mm]	$\sqrt{\frac{\beta_{\min}\epsilon_N}{\beta\gamma}}$	2.36	.35
x ang. spread at QF	$\sigma_{x'}$	$\sqrt{\frac{\epsilon_N}{\beta_{\max}\beta\gamma}}$	$46.1\text{x}10^{-6}$	$6.9\text{x}10^{-6}$
y ang. spread at QF	$\sigma_{y'}$	$\sqrt{\frac{\epsilon_N}{\beta_{\min}\beta\gamma}}$	$89.5\text{x}10^{-6}$	$13.3\text{x}10^{-6}$

Beam Lines

- In our definition and derivation of the lattice function, a closed path through a periodic system. This definition doesn't exist for a beam line, but once we know the lattice functions at one point, we know how to propagate the lattice function down the beam line.

$$\mathbf{M}(out, in)$$

$$\begin{pmatrix}\alpha_{in}\\ \beta_{in}\\ \gamma_{in}\end{pmatrix}$$

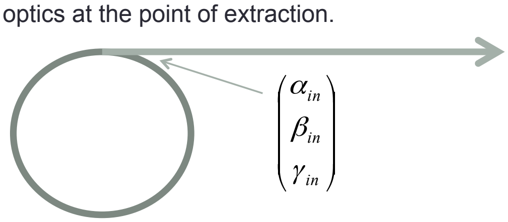


$$\begin{pmatrix}\alpha_{out}\\ \beta_{out}\\ \gamma_{out}\end{pmatrix}$$

$$\begin{pmatrix}\alpha_{out}\\ \beta_{out}\\ \gamma_{out}\end{pmatrix} = \begin{pmatrix}(m_{11}m_{22}+m_{12}m_{21}) & (-m_{11}m_{21}) & (-m_{12}m_{22})\\ (-2m_{11}m_{12}) & (m_{11}^2) & (m_{12}^2)\\ (-2m_{21}m_{22}) & (m_{21}^2) & (m_{22}^2)\end{pmatrix}\begin{pmatrix}\alpha_{in}\\ \beta_{in}\\ \gamma_{in}\end{pmatrix}$$

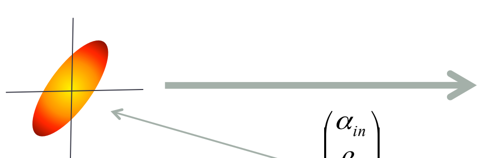
Establishing Initial Conditions

- When extracting beam from a ring, the initial optics of the beam line are set by the optics at the point of extraction.



$$\begin{pmatrix}\alpha_{in}\\ \beta_{in}\\ \gamma_{in}\end{pmatrix}$$

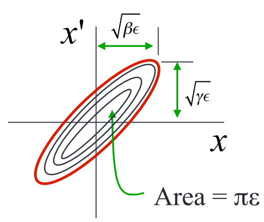
- For particles from a source, the initial lattice functions can be defined by the distribution of the particles out of the source



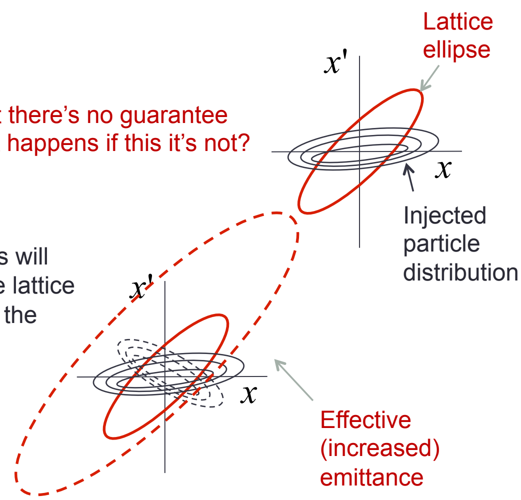
$$\begin{pmatrix}\alpha_{in}\\ \beta_{in}\\ \gamma_{in}\end{pmatrix}$$

Mismatch and Emittance Dilution

- In our previous discussion, we implicitly assumed that the distribution of particles in phase space followed the ellipse defined by the lattice function



…but there's no guarantee
What happens if this it's not?



- Once injected, these particles will follow the path defined by the lattice ellipse, effectively increasing the emittance

Modeling FODO Cells in g4beamline

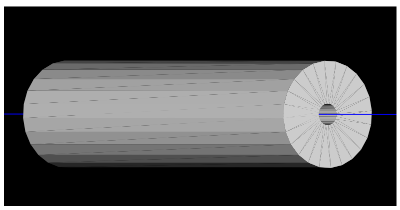
- In spite of the name, g4beamline is not really a beam line tool.
  - Does not automatically handle recirculating or periodic systems
  - Does not automatically determine reference trajectory
  - Does not match or directly calculate Twiss parameters
    - Fits particle distributions to determine Twiss parameters and statistics.
- Nevertheless, it's so easy to use, that we can work around these shortcomings
  - Create a series of FODO cells
  - Carefully match our initial particle distributions to the calculations we just made.

Creating a Line of FODO Cells

- We've calculated everything we need to easily create a string of FODO cells based on the Main Ring
  - Won't worry about bends
  - Phase advance per cell = 71°, so need at least at least ~5 cells to see one betatron period. Let's do 8.
- First, create a quadrupole

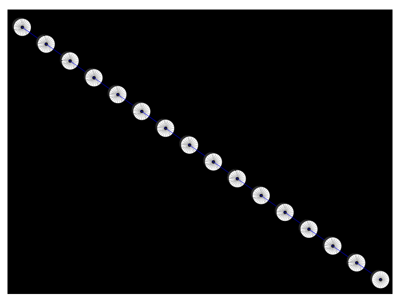
```
# Main Ring FODO cell
param L=29740.
param QL=2133.3
param aperture=50.    # Not really important
param -unset gradient=24.479
param -unset nCell=8
genericquad MRQuad fieldLength=$QL ironLength=$QL apertureRadius=$aperture ironRadius=5*$aperture  kill=1
```

kill=1 saves time if you make a mistake!



Doesn't really look like a real quadrupole, but the field is right and that's all that matters.

Create string of FODOs



- Create 8 cells by putting 16 of these, spaced 29740 mm apart, with *alternating* gradients (good practice for doing loops in g4bl).
  - If we put the first quad at z=0., the beam will start the middle of it. Is this what we want? Why or why not?
- We create a Gaussian beam based on the parameters we calculated on Slide 50

```
beam gaussian sigmaX=.682 sigmaXp=.00000686 sigmaY=.351 sigmaYp=.00001332 \
       meanMomentum=$P nEvents=$nEvents particle=proton
```

- We want to track the first hundred particles individually, so we add

```
trace nTrace=100 oneNTuple=1 primaryOnly=1
```

- We want to fit the distributions at regular intervals to calculate the beam widths and Twiss parameters. so we add the lines

```
param totlen=2*$nCell*$L
profile zloop=0:$totlen:100 particle=proton file=main_ring_profile.txt
```

- Now run 1000 event (need enough for robust fits)

Analyze Output

- The individual track information is written to the standard root output file in an Ntuple called "AllTracks"

```
TFile ft("g4beamline.root");
TNtuple *t = (TNtuple *) ft.FindObjectAny("AllTracks");
```

- The profile information is written to a text file. This can be read directly into histoRoot. I've provided a class (G4BLProfile) to load it into root*

```
G4BLProfile fp(filename);
TNtuple *p = fp.getNtuple();
```

- We want to create a plotting space on which we can overlay several plots. The easiest way I know do to this is an empty 2D histogram.

```
TH2F plot("plot","Track Trajectories",2,0,sMax,2,-xMax,xMax);
plot.SetStats(kFALSE); //turn off annoying stats box
plot.Draw();
```

- Overlay a 3 sigma "envelope", based on the fitted profiles

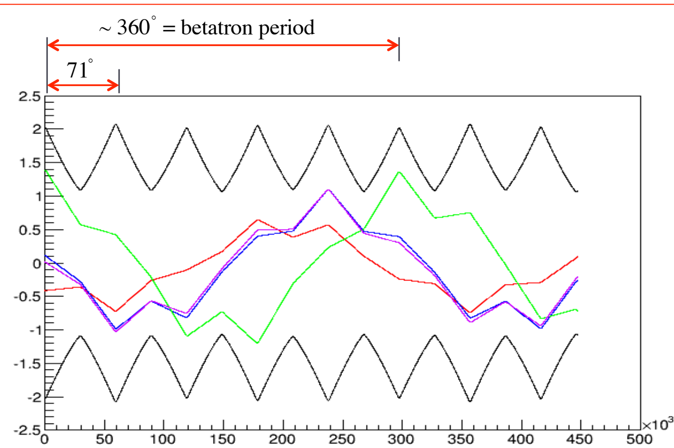
```
p->Draw("3*sigmaX:Z","","same");
p->Draw("-3*sigmaX:Z","","Same");
```

"same" option draws over existing plot

*http://home.fnal.gov/~prebys/misc/NIU_Phys_790/

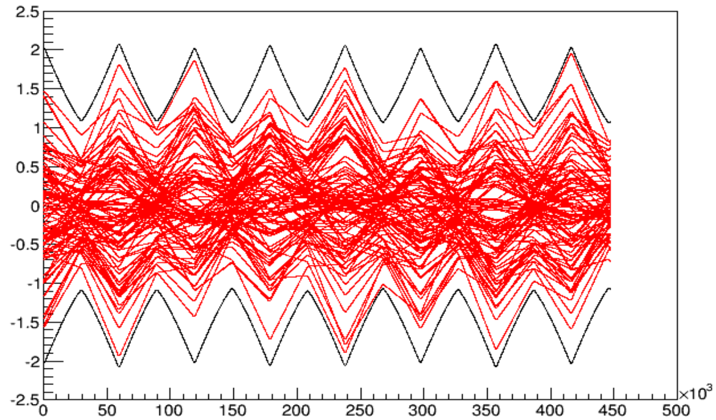
Plot Individual Tracks

```
t->SetMarkerColor(kRed); // These are points, not lines
t->Draw("x:z","EventID==1","same");
...
```



Envelopes

- If we overlay all 100 tracks (remove "EventID" cut), we see that although each track has a periodicity of ~5 cells, the envelope has a periodicity of one cell

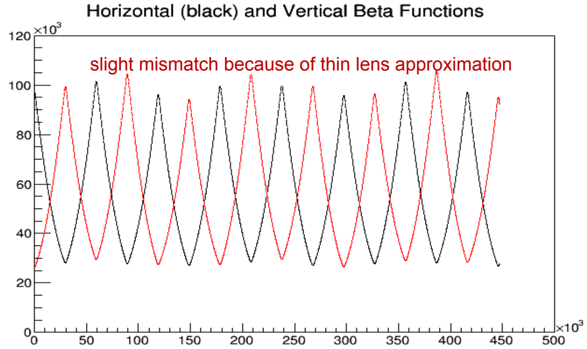


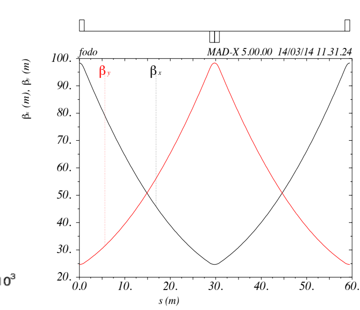
Lattice Functions

- We can plot the fitted lattice functions and compare them to our calculations.

```
TH2F beta("beta","Horizontal (black) and Vertical Beta Functions",2,0.,sMax,2,0.,120000.);
beta.SetStats(kFALSE);
beta.Draw();
p->Draw("betaX:Z","","same");
p->SetMarkerColor(kRed);
p->Draw("betaY:Z","","same");
```

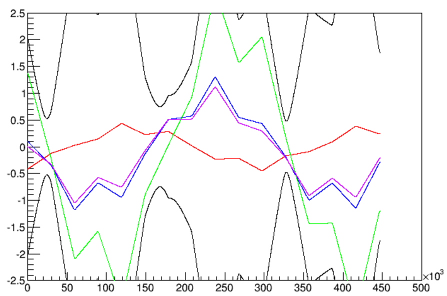
beta functions in mm!



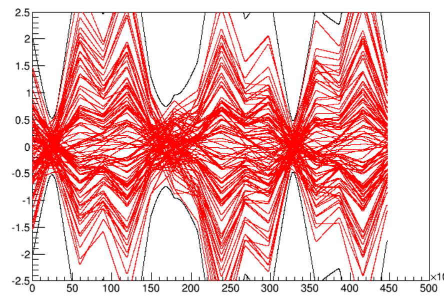


How Important is Matching?

- In our example, we carefully matched our initial distributions to the calculated lattice parameters at the center of the magnet. If we start these same distributions just ~1m upstream, at the entrance to the magnet, things aren't so nice.



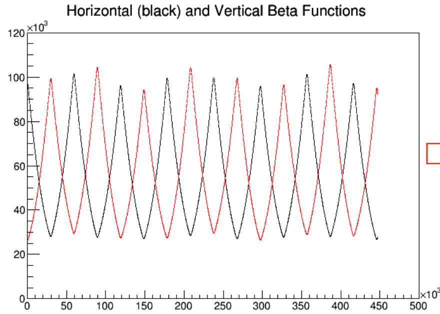
Individual track trajectories look similar (of course)

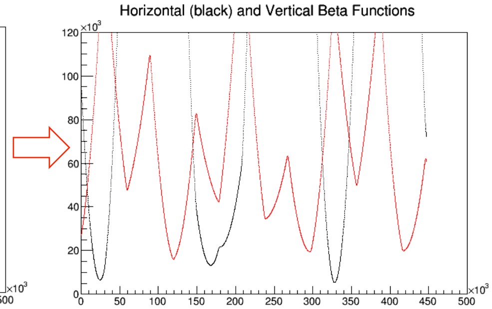


Envelope looks totally crazy

Mismatching and Lattice Parameters

- If we fit the beta functions of these distributions, we see that the original periodicity is completely lost now.





- Therefore, lattice matching is very important when injecting, extracting, or transitioning between different regions of a beam line!