Floquet Coordinates, Resonance, and Couplings

Non-linear Perturbations

- In our earlier lectures, we found the general equations of motion

$$x'' = -\frac{B_y}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2}$$

$$y'' = \frac{B_x}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2$$

- We initially considered only the linear fields, but now we will bundle all additional terms into ΔB
  - non-linear plus linear field errors

$$\begin{aligned} B_y &= B_0 + B'x + \Delta B_y(x,s) \\ B_x &= \underbrace{\phantom{B_0 +{}} B'y}_{\text{This part gave us the Hill's equation}} + \Delta B_x(y,s) \end{aligned}$$

- We see that if we keep the lowest order term in ΔB, we have

$$\boxed{\begin{aligned} x'' + K(s)x &= -\frac{1}{(B\rho)}\Delta B_y(x,s) \\ y'' + K(s)y &= \frac{1}{(B\rho)}\Delta B_x(y,s) \end{aligned}}$$

Floquet Transformation

- Evaluating these perturbed equations can be very complicated, so we will seek a transformation which will simplify things
- Our general equation of Motion is

$$x(s) = A\sqrt{\beta(s)}\cos\left(\psi(s) + \delta\right)$$

- This looks quite a bit like a harmonic oscillator, so not surprisingly there is a transformation which looks *exactly* like harmonic oscillations

$$\xi(s) = \frac{x}{\sqrt{\beta}}$$

$$\phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int \frac{1}{\beta} ds \Rightarrow \frac{d\phi}{ds} = \frac{1}{\nu\beta}$$

- After a couple pages of messy math, we find that the general equation of motion is

Harmonic Oscillator

$$\boxed{\ddot{\xi} + \nu^2\xi = -\nu^2\beta^{3/2}\frac{\Delta B}{(B\rho)}}$$

Driving Term

Understanding Floquet Coordinates

- In the absence of nonlinear terms, our equation of motion is simply that of a harmonic oscillator

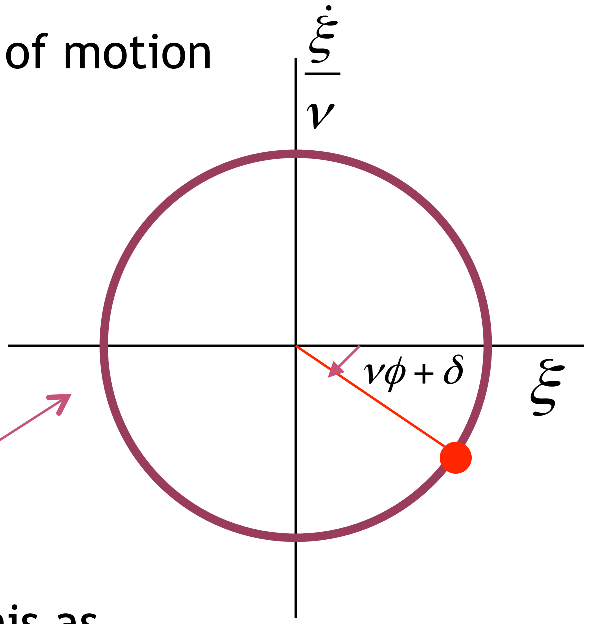
$$\ddot{\xi}(\phi)+\nu^2\xi(\phi)=0$$

and we write down the solution

$$\xi(\phi)=a\cos\left(\nu\phi+\delta\right)$$

$$\dot{\xi}(\phi)=-a\nu\sin\left(\nu\phi+\delta\right)$$

- Thus, motion is a circle in the $\left(\xi,\frac{\dot{\xi}}{\nu}\right)$ plane



- Using our standard formalism, we can express this as

$$\begin{aligned}\xi(\varphi)&=\xi_0\cos\left(\nu\varphi\right)+\frac{\dot{\xi}_0}{\nu}\sin\left(\nu\varphi\right)\\ \dot{\xi}(\varphi)&=-\xi_0\nu\sin\left(\nu\varphi\right)+\dot{\xi}_0\cos\left(\nu\varphi\right)\end{aligned}\Rightarrow\begin{pmatrix}\xi(\varphi)\\ \dot{\xi}(\varphi)\end{pmatrix}=\begin{pmatrix}\cos\left(\nu\varphi\right) & \tilde{\beta}\sin\left(\nu\varphi\right)\\ -\frac{1}{\tilde{\beta}}\sin\left(\nu\varphi\right) & \cos\left(\nu\varphi\right)\end{pmatrix}\begin{pmatrix}\xi_0\\ \dot{\xi}_0\end{pmatrix};\ \text{where } \tilde{\beta}\equiv\frac{1}{\nu}$$

- A common mistake is to view ϕ as the phase angle of the oscillation.
  - $\nu\varphi$ the phase angle of the oscillation
  - φ advances by 2π in one revolution, so it's *related* (but NOT equal to!) the angle around the ring.

$$\text{Note: } x_{\max}^2=\beta\varepsilon=\beta\xi_{\max}^2=\beta a^2\Rightarrow a^2=\epsilon$$

unnormalized!

Perturbations

- In general, resonant growth will occur if the perturbation has a component at the same frequency as the unperturbed oscillation; that is if

$$\Delta B(\xi,\phi) = ae^{i\nu\phi} + (\ldots) \Rightarrow \text{resonance!}$$

- We will expand our magnetic errors at one point in ϕ as

$$\Delta B(x) \equiv b_0 + b_1 x + b_2 x^2 + b_3 x^3 \ldots; b_n \equiv \frac{1}{n!}\frac{\partial^n B}{\partial x^n}\bigg|_{x=y=0}$$

Note:

$$\boxed{\begin{aligned} b_n &= b_n(s) \\ &= b_n(\phi) \end{aligned}}$$

$$\boxed{x = \sqrt{\beta}\xi}$$

$$-\frac{\nu^2\beta^{3/2}\Delta B}{(B\rho)} = -\frac{\nu^2}{(B\rho)}\left(\beta^{3/2}b_0 + \beta^{4/2}b_1\xi + \beta^{5/2}b_2\xi^2 + \ldots\right)$$

$$\ddot{\xi} + \nu^2\xi = -\frac{\nu^2}{(B\rho)}\sum_{n=0}^{\infty}\beta^{(n+3)/2}b_n\xi^n$$

- But in general, b_n is a function of φ, as is β, so we bundle all the dependence into harmonics of φ

$$\frac{1}{(B\rho)}\beta^{(n+3)/2}b_n = \sum_{m=-\infty}^{\infty}C_{m,n}e^{im\phi}$$

- So the equation associated with the n^th driving term becomes

$$\ddot{\xi} + \nu^2\xi = -\nu^2\sum_{k=-\infty}^{\infty}C_{m,n}\xi^n e^{im\phi}$$

Remember!
ξ,β, and b_n are all functions of (only) φ

Calculating Driving Terms

$$\boxed{\int_0^{2\pi} e^{in\phi} e^{-im\phi}\, d\phi = 2\pi\delta_{m,n}}$$

- We can Fourier transform to calculate the $C_{m,n}$ coefficients based on the measured fields

$$C_{m,n} = \frac{1}{(B\rho)} \frac{1}{2\pi} \int_0^{2\pi} \beta^{(n+3)/2} b_n e^{-im\phi} d\phi$$

- But we generally know things as functions of *s*, so we use $d\phi = \frac{1}{\nu} d\psi = \frac{1}{\nu} \frac{d\psi}{ds} ds = \frac{1}{\nu\beta} ds$ to get

$$\boxed{C_{m,n} = \frac{1}{(B\rho)} \frac{1}{2\pi\nu} \oint \beta^{(n+1)/2}(s) b_n(s) e^{-im\phi}\, ds}$$

Where (for a change) we have explicitly shown the *s* dependent terms.

- We're going to assume small perturbations, so we can approximate β with the solution to the homogeneous equation

$$\ddot{\xi} + \nu^2 \xi = -\nu^2 \sum_{k=-\infty}^{\infty} C_{m,n} \xi^n e^{im\phi}$$

$$\xi(\varphi) \approx a\cos(\nu\varphi); \ \ \text{(define starting point so } \delta = 0\text{)}$$

$$\xi^{\mathrm{n}} = a^n \cos^n(\nu\varphi) = \mathrm{Re}\left[a^n \frac{1}{2^n} \sum_{\substack{k=-n \\ \Delta k=2}}^{n} \begin{pmatrix} n \\ \frac{n-k}{2} \end{pmatrix} e^{i\nu k\phi} \right]; \text{ where } \begin{pmatrix} \mathrm{i} \\ \mathrm{j} \end{pmatrix} \equiv \frac{i!}{j!(i-j)!}$$

- Example

$$\cos^3\theta = \frac{1}{2^3}\left(\binom{3}{3}\cos(-3\theta)+\binom{3}{2}\cos(-\theta)+\binom{3}{1}\cos(\theta)+\binom{3}{0}\cos(3\theta)\right)=\frac{3}{4}\cos 3\theta+\frac{1}{4}\cos\theta$$

- Plugging this in, we can write the nth driving term as

$$-\nu^2\left(\frac{a}{2}\right)^n \sum_{\substack{k=-n \\ \Delta k=2}}^{n} \binom{n}{\frac{n-k}{2}} \sum_{m=-\infty}^{\infty} C_{m,n}\, e^{i(m+\nu k)\phi} \qquad \binom{\mathrm{i}}{\mathrm{j}} \equiv \frac{i!}{j!(i-j)!}$$

- We see that a resonance will occur whenever

$$\begin{aligned} m+\nu k &= \pm\nu \\ \nu(1\mp k) &= \pm m \end{aligned} \qquad \text{where} \qquad \begin{aligned} &-\infty < m < \infty \\ &-n \le k \le n \quad (\Delta k = 2) \end{aligned}$$

- Since *m* and *k* can have either sign, we can cover all possible combinations by writing

$$\boxed{\nu_{\text{resonant}} = \frac{m}{1-k}}$$

- Reminder
  - n= power of multipole expansion (quad=1, sextupole=2, octupole=2, etc)
  - m= Fourier component of anomalous magnetic component when integrated around the ring.

Types of Resonances

Magnet Type	n	k	Order $\lvert 1-k \rvert$	Resonant tunes $\nu=m/(1-k)$	Fractional Tune at Instability
Dipole	0	0	1	m	0,1
Quadrupole	1	1	0	*none (tune shift)*	-
	1	-1	2	$m/2$	0,1/2,1
Sextupole	2	2	1	m	0,1
	2	0	1	m	0,1
	2	-2	3	$m/3$	0,1/3,2/3,1
Octupole	3	3	2	$m/2$	0,1/2,1
	3	1	0	*None*	-
	3	-1	2	$m/2$	0,1/2,1
	3	-3	4	$m/4$	0,1/4,1/2,3/4,1

Example: Sextupole (Third Order Resonance)

- The third order resonance will occur at tunes near m/3.
- The strength of the resonance will be given by

Sextupole term

Convert back to ordinary phase angle

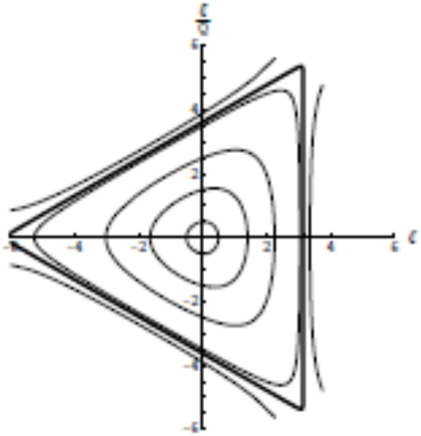
$$A_{m,2} = \oint \beta^{3/2} \frac{B''}{2(B\rho)} \cos(3\psi) ds$$

$$B_{m,2} = \oint \beta^{3/2} \frac{B''}{2(B\rho)} \sin(3\psi) ds$$

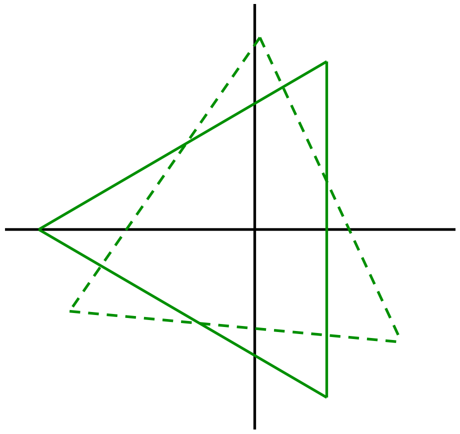
- It will perturb the stable region of phase space into a triangle

$A_{m,2} \neq 0$

$B_{m,2} = 0$



Relative size of Terms determine Orientation in phase space



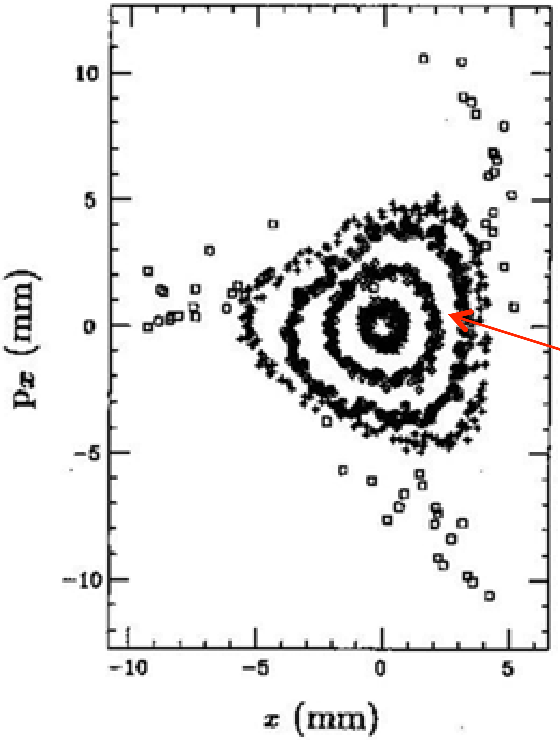
Strength of Resonance

- The size of the stable region in phase space with shrink with increased driving strength or by moving the tune closer to m/3.

$$\delta\nu = \nu - \frac{m}{3}$$

$$A_{m,2} = \oint \beta^{3/2} \frac{B''}{2(B\rho)} \cos(3\psi) ds \quad [\mathrm{L}]^{-1/2}$$

$$B_{m,2} = \oint \beta^{3/2} \frac{B''}{2(B\rho)} \sin(3\psi) ds \quad [\mathrm{L}]^{-1/2}$$

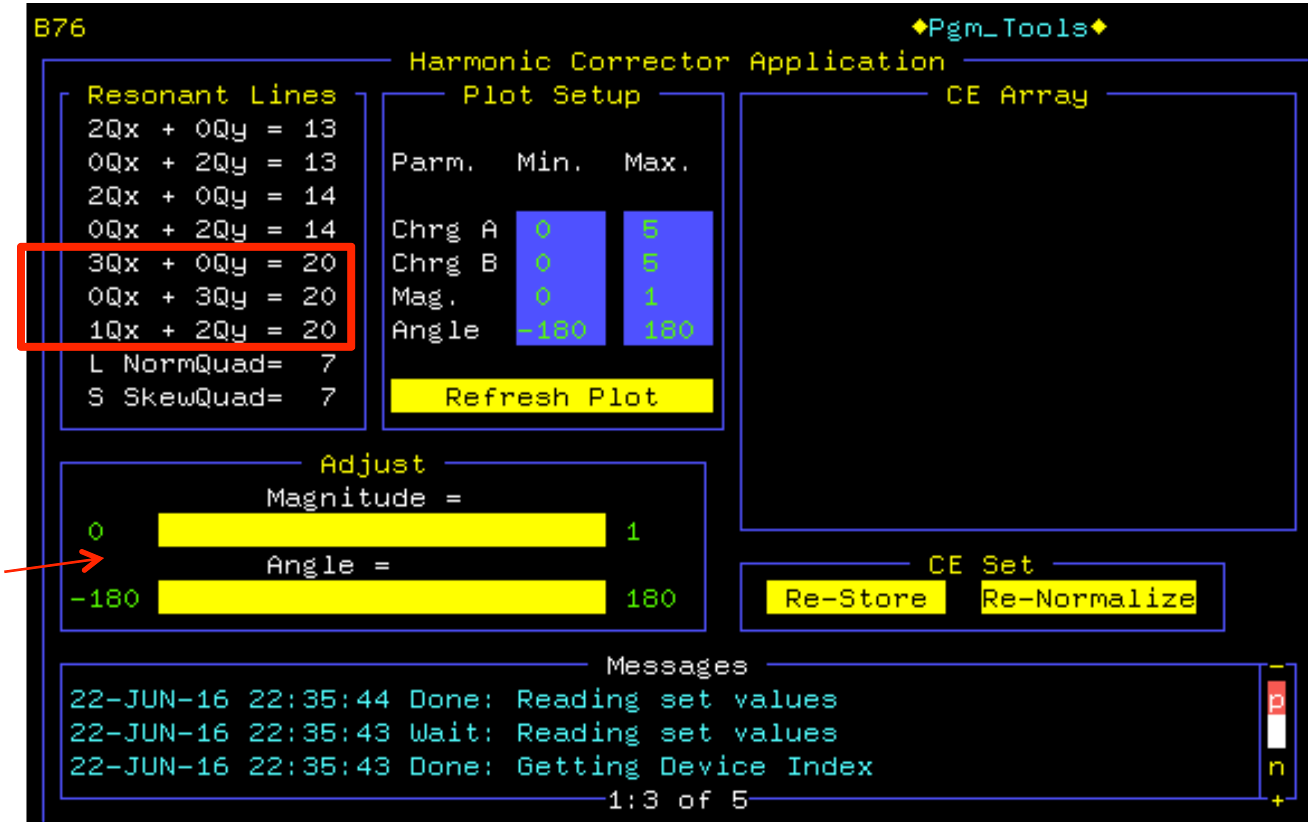


$$\varepsilon_{\max} = \frac{64\pi^2\delta\nu^2}{3\left(A_{m,2}^2 + B_{m,2}^2\right)}$$

$$\delta\nu = \frac{\sqrt{3\varepsilon\left(A_{m,2}^2 + B_{m,2}^2\right)}}{8\pi}$$

Use of Driving Terms

- We see that in general, magnetic imperfections drive specific resonances based on the harmonic content of individual multipole moments.
  - This means that unwanted resonances can be canceled out by installing multipoles to drive that specific harmonic term.
  - Example: Booster harmonic corrector page



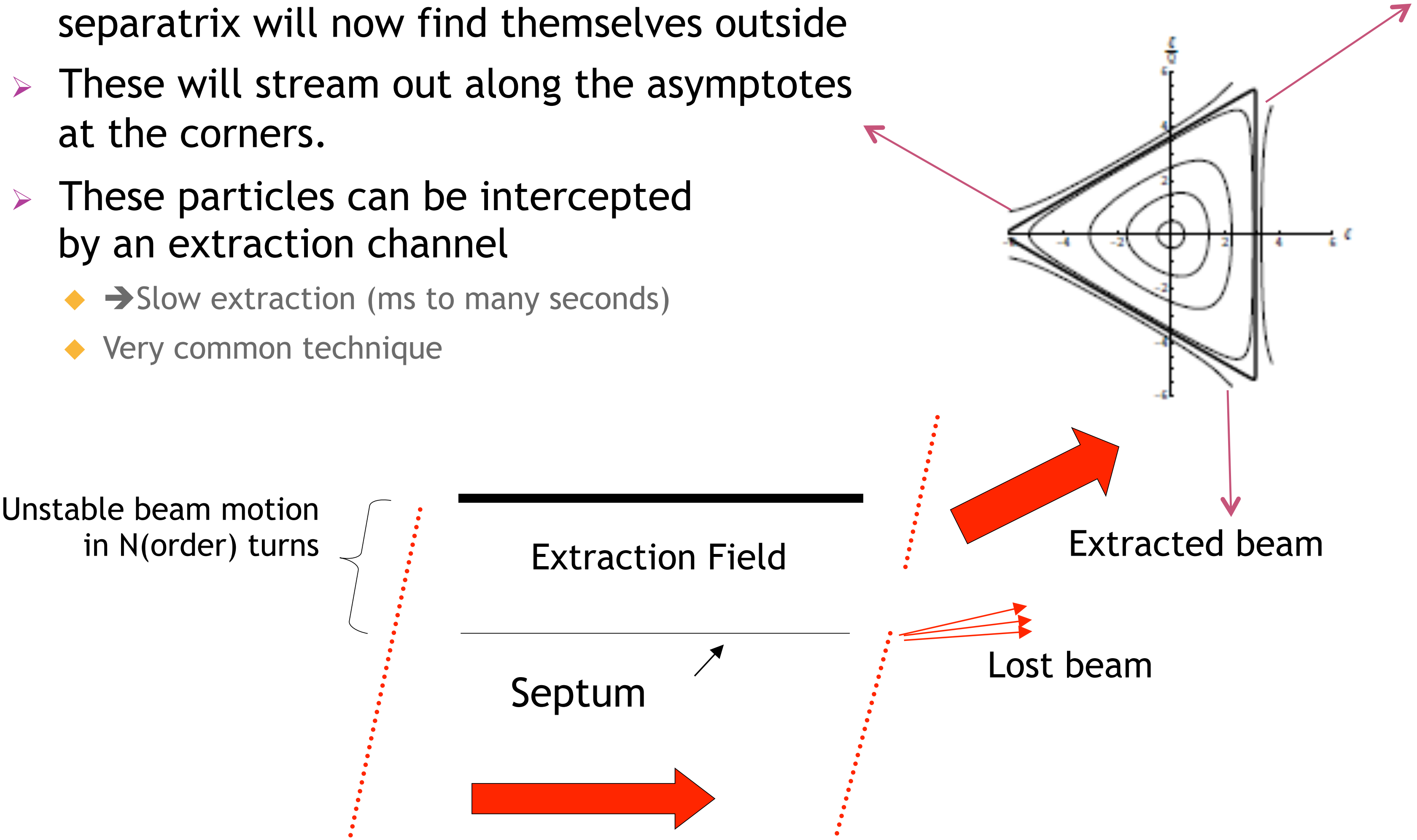
```
B76                                           ♦Pgm_Tools♦
                  Harmonic Corrector Application

Resonant Lines          Plot Setup                 CE Array
2Qx + 0Qy = 13
0Qx + 2Qy = 13          Parm.   Min.   Max.
2Qx + 0Qy = 14
0Qx + 2Qy = 14          Chrg A  0      5
3Qx + 0Qy = 20          Chrg B  0      5
0Qx + 3Qy = 20          Mag.    0      1
1Qx + 2Qy = 20          Angle   -180   180
L  NormQuad=   7
S  SkewQuad=   7

              Adjust
        Magnitude =
0                          1
        Angle =
-180                       180

                            Messages
22-JUN-16 22:35:44 Done: Reading set values
22-JUN-16 22:35:43 Wait: Reading set values
22-JUN-16 22:35:43 Done: Getting Device Index
                            1:3 of 5
```

Third order resonances near ν=6 2/3

Control magnitude and phase (A vs B terms)

Application of Resonance

- If we increase the driving term (or move the tune closer to m/3), then the area of the triangle will shrink, and particles which were inside the separatrix will now find themselves outside
- These will stream out along the asymptotes at the corners.
- These particles can be intercepted by an extraction channel
  - ➔Slow extraction (ms to many seconds)
  - Very common technique



Unstable beam motion in N(order) turns

Extraction Field

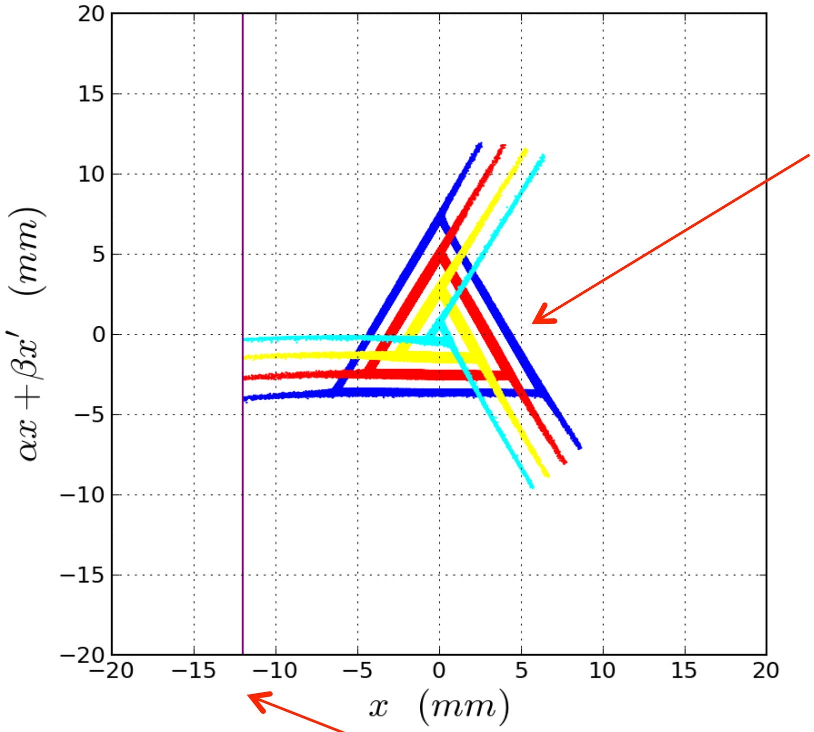
Septum

Extracted beam

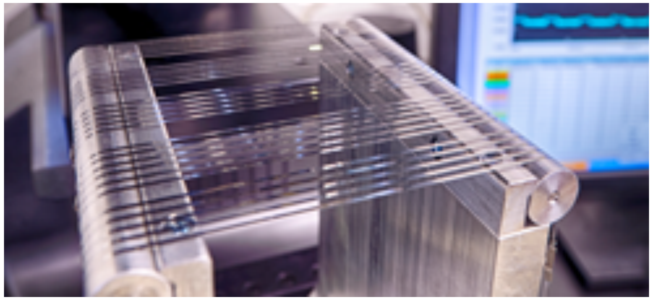
Lost beam

Example: Mu2e Experiment 8 GeV Extraction

- Use sextupoles to drive 3rd integer resonance

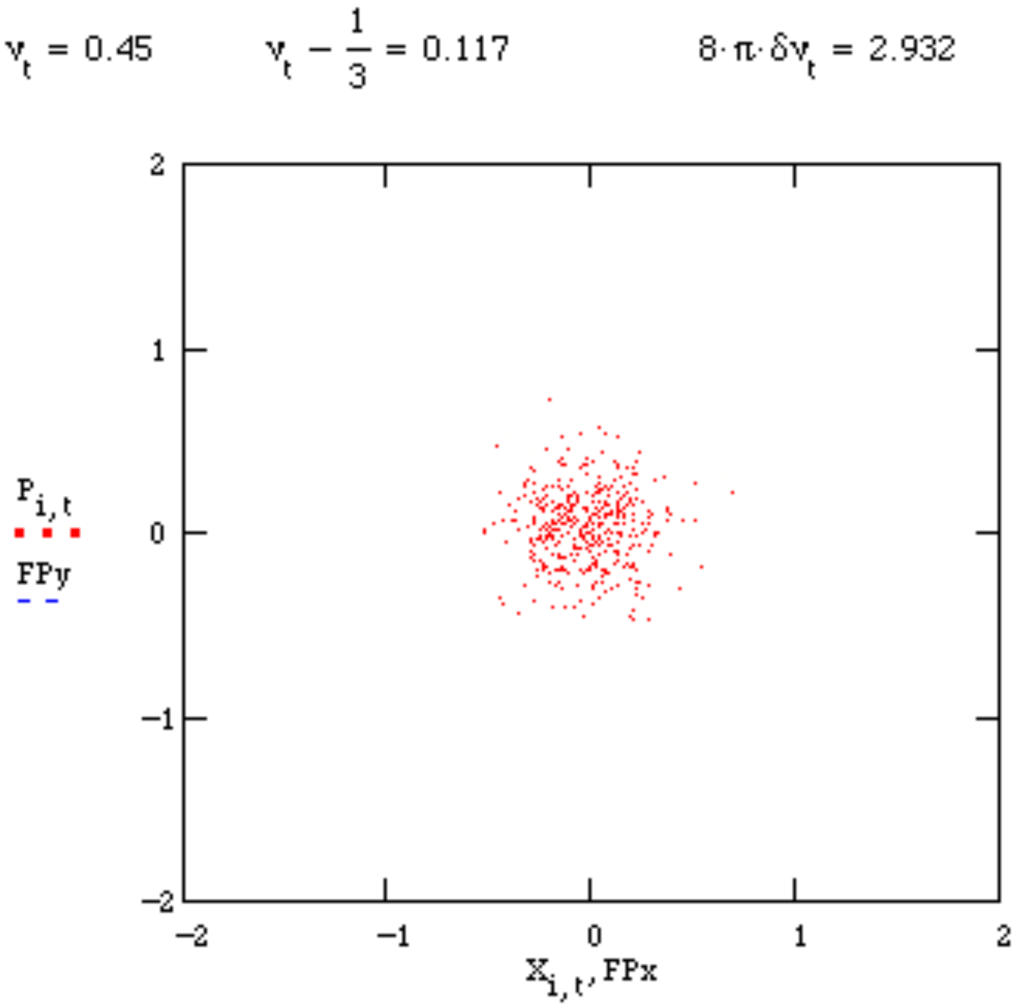


Moving tune closer to m/3 will reduce stable phase space, causing beam to be removed at a steady rate



Electrostatic septum at 80 kV/1cm deflects beam into a downstream Lambertson magnet

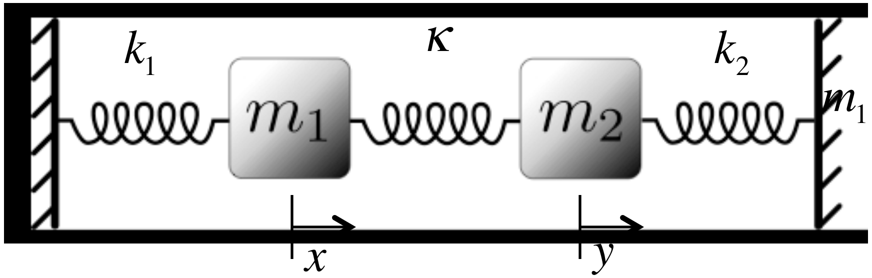
Simulation of Third Integer Extraction*



$\nu_t = 0.45$ $\nu_t - \frac{1}{3} = 0.117$ $8 \cdot \pi \cdot \delta\nu_t = 2.932$

*M. Syphers

Coupled Harmonic Oscillators



$m_1 = m_2 = m$

Equations of motion

$$m\ddot{x} + k_1 x + \kappa(x - y) = 0$$
$$m\ddot{x} + (k_1 + \kappa)x - \kappa y = 0$$
$$m\ddot{y} + (k_2 + \kappa)y - \kappa x = 0$$

Define uncoupled frequencies:

$$\omega_1^2 \equiv \frac{(k_1 + \kappa)}{m}; \omega_2^2 \equiv \frac{(k_2 + \kappa)}{m}; q^2 \equiv \frac{\kappa}{m}$$

$$\ddot{x} + \omega_1^2 x - q^2 y = 0$$
$$\ddot{y} + \omega_2^2 y - q^2 x = 0$$

Try a solution of he form:

$$x = ae^{i\omega t} \Rightarrow \ddot{x} = -\omega^2 ae^{i\omega t}$$
$$y = be^{i\omega t} \Rightarrow \ddot{y} = -\omega^2 be^{i\omega t}$$

$$a\left(\omega_1^2 - \omega^2\right)e^{i\omega t} = bq^2 e^{i\omega t}$$
$$b\left(\omega_2^2 - \omega^2\right)e^{i\omega t} = aq^2 e^{i\omega t}$$

Multiply the top by the bottom:

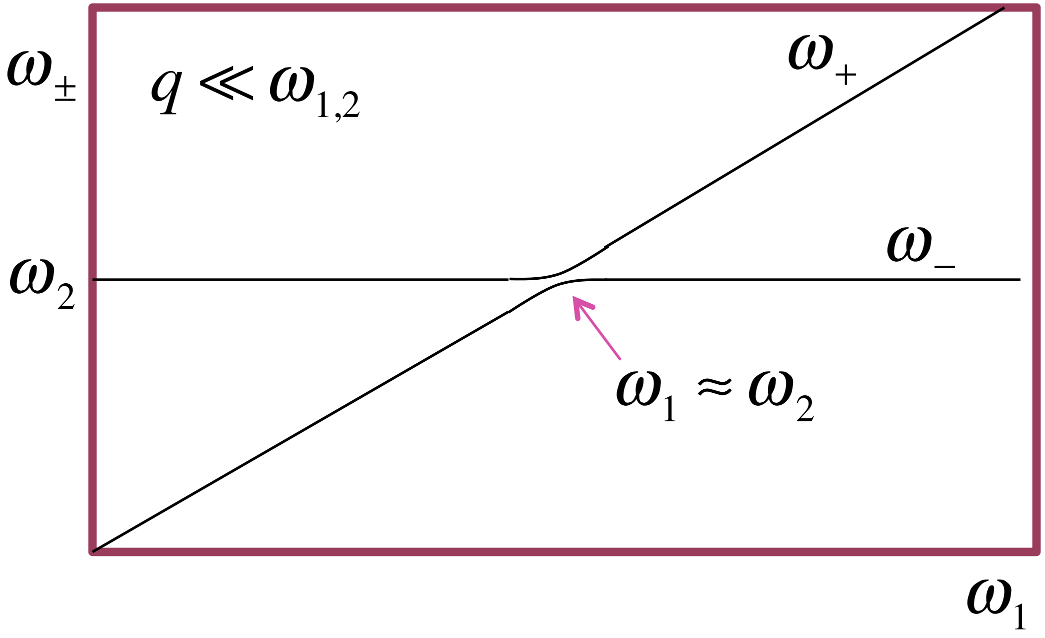
$$\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right) = q^4$$

$$\omega^4 - \left(\omega_1^2 + \omega_2^2\right)\omega^2 + \left(\omega_1^2\omega_2^2 - q^4\right) = 0$$

$$\rightarrow \quad \omega^2 = \frac{\left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 4\omega_1\omega_2 + 4q^4}}{2}$$

$$\boxed{= \frac{\left(\omega_1^2 + \omega_2^2\right) \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + 4q^4}}{2}}$$

Weak Coupling



Degenerate Case:

$$\omega_1 = \omega_2 \equiv \omega_0$$

$$\omega^2 = \omega_0^2 \pm q^2 = \frac{k_0}{m} \pm \frac{\kappa}{m}$$

Resonance splitting

Formalism

General coupled equation

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{i\omega t} \to \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad \left(-\omega^2 \mathbf{I} + \mathbf{M}\right) \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

- i.e. ω^2 are the eigenvalues of **M** and (a,b) are the linear combinations of x and y which undergo simple harmonic motion.

Application to Accelerators

Introduce skew-quadrupole term

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} \neq 0$$

$$x' \propto -\frac{\partial B_y}{\partial x} x - \frac{\partial B_y}{\partial y} y$$

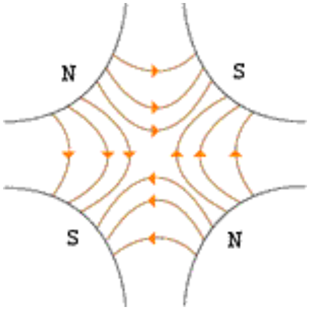
$$y' \propto \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x$$

Planes coupled
x and y motion *not* independent

General Transfer Matrix

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix}$$

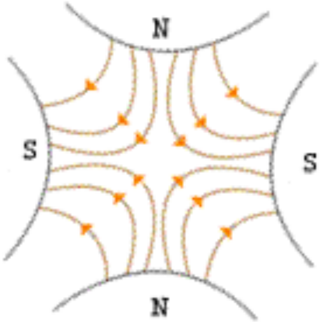
Normal Quad



$$\frac{1}{f} \equiv q = \frac{B'l}{(B\rho)}$$

$$\mathbf{M_Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q & 1 \end{pmatrix}$$

Skew quad



$$B_x = \tilde{B}'x \rightarrow \Delta y' = \frac{\tilde{B}'l}{(B\rho)}x \equiv \tilde{q}x$$

$$B_y = -\tilde{B}'y \rightarrow \Delta x' = \frac{\tilde{B}'l}{(B\rho)}y \equiv \tilde{q}y$$

So the transfer matrix for a skew quad would be:

$$\mathbf{M}_{\tilde{\mathbf{Q}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \tilde{q} & 0 \\ 0 & 0 & 1 & 0 \\ \tilde{q} & 0 & 0 & 1 \end{pmatrix}$$

For a normal quad rotated by φ it would be

$$\mathbf{M}_{\mathbf{Q}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q\cos 2\phi & 1 & -q\sin 2\phi & 0 \\ 0 & 0 & 1 & 0 \\ -q\sin 2\phi & 0 & q\cos 2\phi & 1 \end{pmatrix}$$

Coupling in Floquet Coordinates

$$\Delta x' = \tilde{q} y$$

$$\Delta x = 0 = \sqrt{\beta_x}\Delta\xi_x$$

In Floquet Coordinates

$$x' = \frac{1}{\sqrt{\beta_x \nu_x}}\left(\dot{\xi}_x - \alpha_x \nu_x \xi_x\right)$$

$$\Delta x' = \frac{\Delta\dot{\xi}_x}{\sqrt{\beta_x \nu_x}} = \tilde{q} y = \tilde{q}\sqrt{\beta_y}\xi_y$$

$$\Rightarrow$$

$$\Delta\dot{\xi}_x = \nu_x \kappa \xi_y$$

$$\Delta\dot{\xi}_y = \nu_y \kappa \xi_x$$

where $\boxed{\kappa \equiv \tilde{q}\sqrt{\beta_x \beta_y}}$

Dimensionless coupling

Coupled Tunes

$$\bar{\nu} \equiv \frac{(\nu_x + \nu_y)}{2}$$

$$\delta\nu = \nu_y - \nu_x$$

$$\kappa \equiv \tilde{q}\sqrt{\beta_x \beta_y}$$

$$\nu_\pm = \bar{\nu} \pm \frac{\delta\nu}{2}\sqrt{1 + \frac{\kappa^2}{4\pi^2\delta\nu^2}}$$

$$= \bar{\nu} \pm \frac{1}{4\pi}\sqrt{4\pi^2\delta\nu^2 + \kappa^2}$$

If there’s no coupling, then

$$\nu_\pm = \bar{\nu} \pm \frac{\delta\nu}{2}$$

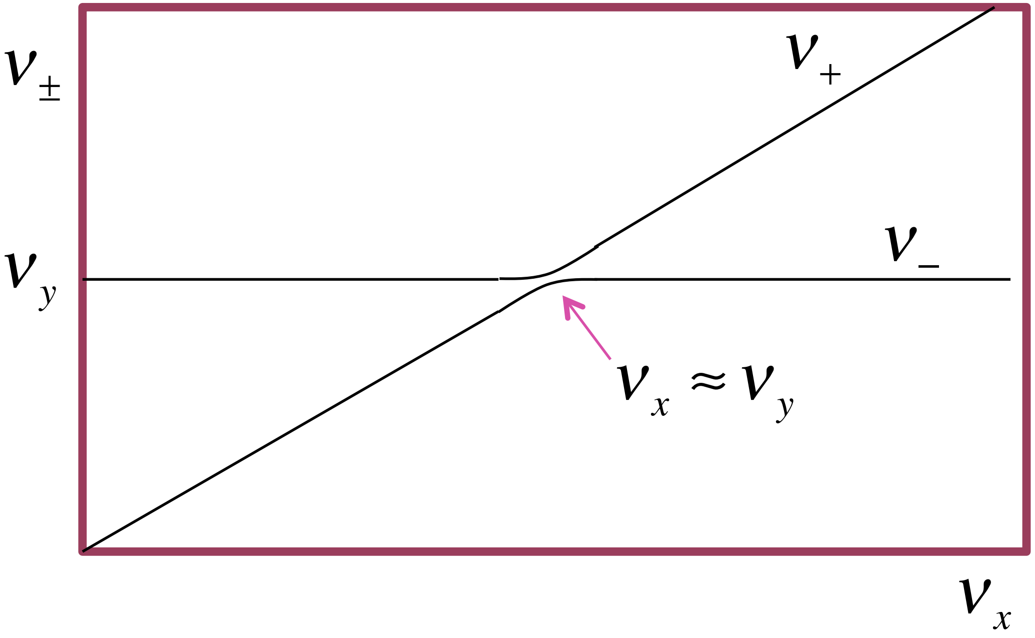
$$= \nu_{x,y}$$

If there’s coupling, then there will always be a tune split

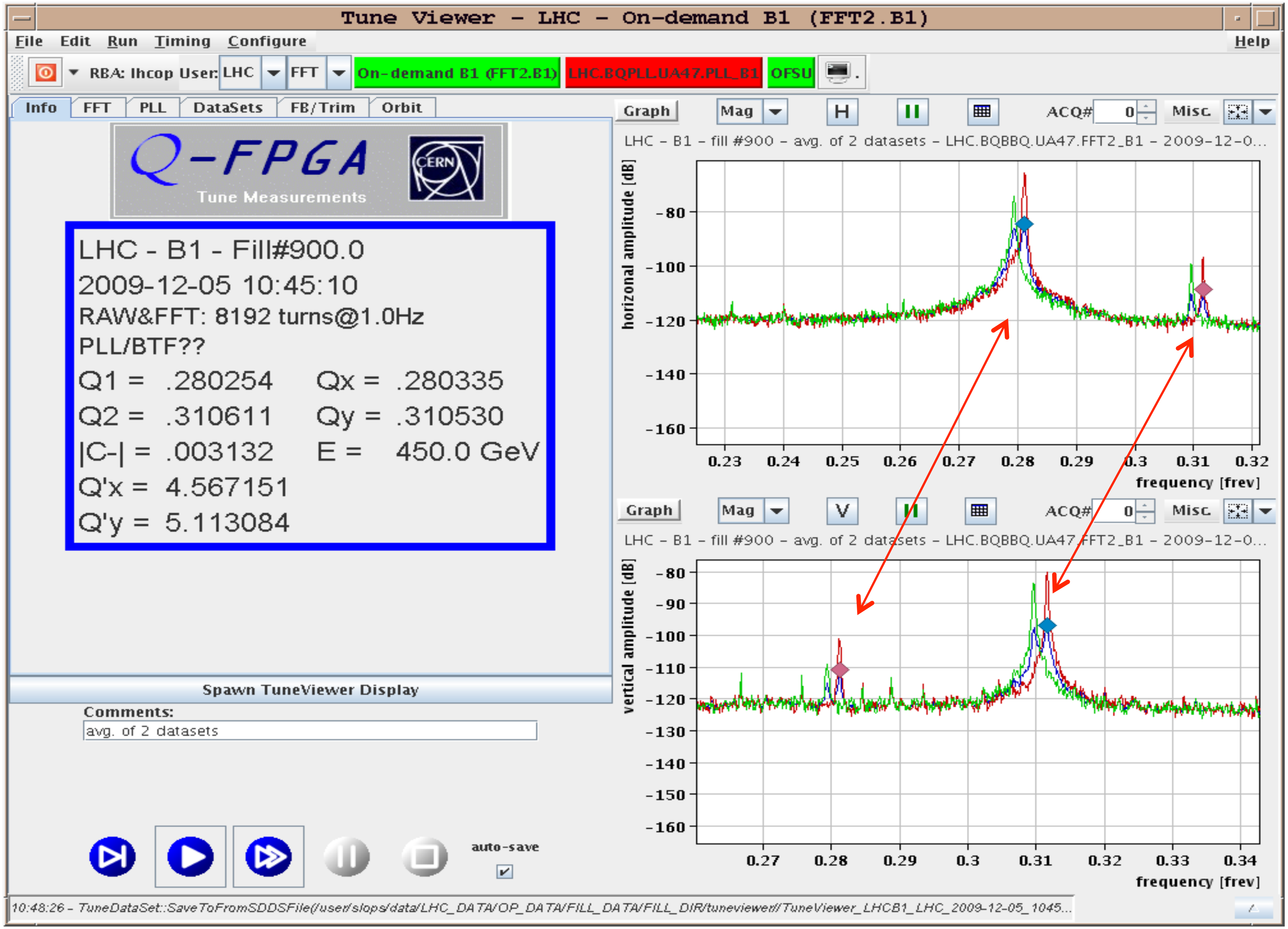
$$\nu_x = \nu_y = \nu$$

$$\rightarrow \delta\nu = 0$$

$$\boxed{\begin{aligned}\Delta\nu_{min} &= \nu_+ - \nu_- \\ &= \frac{\kappa}{2\pi} = \frac{\sqrt{\beta_x\beta_y}}{2\pi}\tilde{q}\end{aligned}}$$



Example: Tune Coupling in LHC



LHC - B1 - Fill#900.0
2009-12-05 10:45:10
RAW&FFT: 8192 turns@1.0Hz
PLL/BTF??
Q1 = .280254 Qx = .280335
Q2 = .310611 Qy = .310530
|C-| = .003132 E = 450.0 GeV
Q'x = 4.567151
Q'y = 5.113084

RBA: lhcop User: LHC

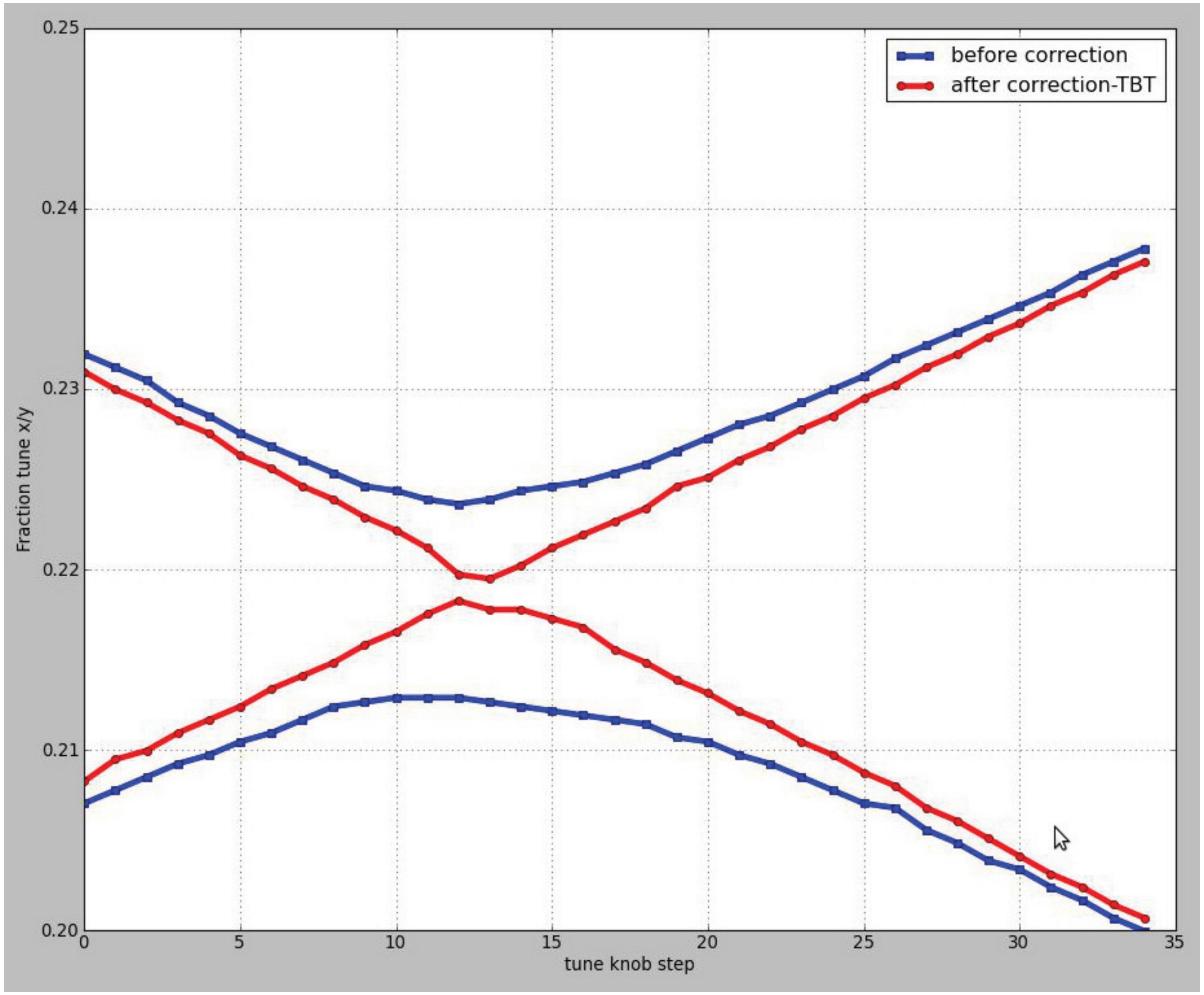
Comments: avg. of 2 datasets

auto-save ☑

```
10:48:26 - TuneDataSet::SaveToFromSDDSFile(/user/slops/data/LHC_DATA/OP_DATA/FILL_DATA/FILL_DIR/tuneviewer//TuneViewer_LHCB1_LHC_2009-12-05_1045...
```

Example: NSLS-II 3 GeV Electron Ring (BNL)

- Minimum tune separation before and after correction.



Example Problem (from 2012 final)

- A 10 GeV proton synchrotron is found to have a minimum tune separation of .02. If this is due to coupling, determine the dimensionless coupling strength κ

$$\Delta\nu_{min} = \frac{\kappa}{2\pi}$$

$$\kappa = (2\pi)\Delta\nu_{min}$$

$$= (2\pi)(.02)$$

$$\boxed{= .126}$$

- If a skew quadrupole is located at a location where bx=50m and by=20, what integrated skew quadrupole field would be required to generate this tune splitting?

$$\kappa = \sqrt{\beta_x\beta_y}\tilde{q}; \quad \tilde{q} = \frac{\tilde{B}'L}{(B\rho)}$$

$$\tilde{B}'L = \frac{(B\rho)\kappa}{\sqrt{\beta_x\beta_y}} = \frac{(36.3)(.126)}{\sqrt{(20)(50)}} \boxed{= .144\text{ T}}$$

(cont'd)

- By what angle would a 1 m long 1 T/m *normal* quad have to be rotated to produce this coupling?

$$\tilde{q} = q \sin 2\phi$$

$$\phi = \frac{1}{2}\sin^{-1}\left(\frac{\tilde{B}'l}{B'l}\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{(.144)}{(1)(1)}\right)$$

$$= .072$$

$$\boxed{= 4.1^\circ}$$

Coupling and Resonances

Although we won't derive it in detail, it's clear that if motion is coupled, we can analyze the system in terms of the normal coordinates, and repeat the analysis in the last chapter. In this case, the normal tunes will be linear combinations of the tunes in the two planes, and so the general condition for resonance becomes.

$$\boxed{k_x \nu_x \pm k_y \nu_y = m \quad (k_x, k_y, m \text{ all integers})}$$

This appears as a set of crossing lines in the nx,ny "tune space". The width of individual lines depends on the details of the machine, and one tries to pick a "working point" to avoid the strongest resonances.