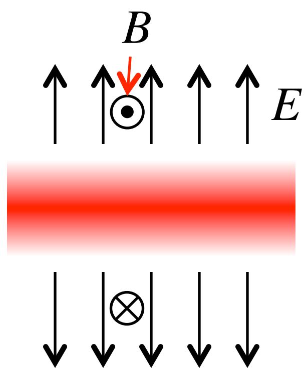
Space Charge and Beam Beam Effects

Space Charge

So far, we have not considered the effect that particles in a bunch might have on each other, or on particles in another bunch.

Consider the effect off space charge on the transverse distribution of the beam.



$$\rho(r)=\frac{Ne}{2\pi\sigma^2}e^{-r^2/2\sigma^2}$$

radial charge density

If we look at the field at a radius r, we have

$$\oint \vec{E}\bullet d\vec{A}=E(2\pi rL)=\frac{Q_{encl}}{\epsilon_0}=\frac{Ne}{\sigma^2}\int_0^r re^{-r^2/2\sigma^2}\,dr$$

$$=Ne\left(1-e^{-r^2/2\sigma^2}\right)$$

$$\longrightarrow \vec{E}=\frac{Ne}{2\pi\epsilon_0 rL}\left(1-e^{-r^2/2\sigma^2}\right)\hat{r}$$

Similarly, Ampere's Law gives

$$\oint \vec{B} \bullet d\vec{l} = 2\pi r B = \mu_0 I_{enclosed} = \mu_0 \frac{Nev}{\sigma^2 L} \int_0^r r e^{-r^2/2\sigma^2}\, dr$$

$$\longrightarrow \vec{B} = \mu_0 \frac{Nev}{2\pi r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{\theta}$$

$$\longrightarrow \vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$= \frac{Ne^2}{2\pi L}\left(1 - e^{-r^2/2\sigma^2}\right)\left(\frac{1}{\epsilon_0}\hat{r} + v^2 \mu_0 \left(\hat{s} \times \hat{\theta}\right)\right)$$

$\hat{s} \times \hat{\theta} = -\hat{r}$

$$\mu_0 = \frac{1}{\epsilon_0}\left(\epsilon_0 \mu_0\right) = \frac{1}{\epsilon_0}\frac{1}{c^2}$$

$$= \hat{r}\frac{Ne^2}{2\pi r L \epsilon_0}\left(1 - e^{-r^2/2\sigma^2}\right)\left(1 - \beta^2\right)$$

$$= \hat{r}\frac{ne^2}{2\pi r \epsilon_0 \gamma^2}\left(1 - e^{-r^2/2\sigma^2}\right); \quad \mathrm{n} \equiv \frac{N}{L} = \frac{dN}{ds}$$

Linear charge density

We can break this into components in x and y

$$F_x = |F|\frac{x}{r}$$

$$= \frac{ne^2}{2\pi\epsilon_0\gamma^2}\frac{x}{r^2}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$= \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{x}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

$$F_y = \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{y}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

Non-linear and coupled ➔ ouch! but for $x \ll \sigma_x$

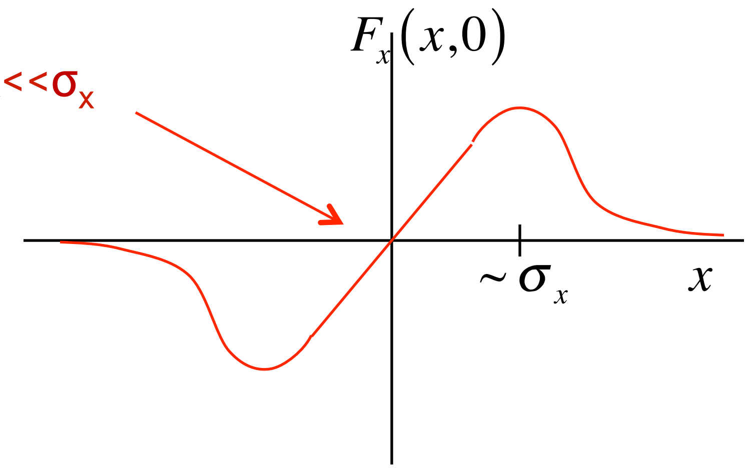
$$\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right) \approx \frac{\left(x^2+y^2\right)}{2\sigma^2}$$

→

$$F_x \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}x$$

$$F_y \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}y$$

~linear and decoupled



$$x'' = \frac{F_x}{vp} = \frac{F_x}{\beta^2 \gamma m c^2}$$

$$\approx \frac{e^2}{4\pi\sigma^2 \epsilon_0 \beta^2 \gamma^3} nx$$

$$= \frac{r_0}{\beta^2 \gamma^3 \sigma^2} nx; \quad r_0 \equiv \frac{e^2}{4\pi\epsilon_0 m_0 c^2}$$

"classical radius" $= 1.53 \times 10^{-18}$ m for protons

This looks like a distributed defocusing quad of strength

$$\frac{d\left(\frac{1}{f}\right)}{ds} \equiv k = -\frac{n r_0}{\beta^2 \gamma^3 \sigma^2}$$

so the total tuneshift is

$$\Delta\nu_x = \frac{1}{4\pi} \oint k \beta_x(s) ds$$

$$= -\frac{r_0}{4\pi\beta^2\gamma^3} \oint n \frac{\beta_x(s)}{\sigma_x^2} ds$$

$\frac{\beta_x(s)}{\sigma_x^2} = \frac{1}{\epsilon_x}$

$$= -\frac{r_0}{4\pi\beta^2\gamma^3} \frac{NB}{\epsilon_x}; \quad \mathrm{B} \equiv \frac{n_{peak}}{\langle n \rangle}$$

"Bunching factor"

$$= -\frac{NBr_0}{4\pi\beta\gamma^2 L} \frac{L}{(\beta\gamma\epsilon_x)}$$

$\beta\gamma\epsilon_x \equiv \epsilon_{x,N}$

$$= -\frac{NBr_0}{4\pi\beta\gamma^2 \epsilon_{x,N}}$$

Maximum tuneshift for particles near core of beam

Example: Fermilab Booster@Injection

$$K = 400 \text{ MeV}$$

$$\text{N}=5\times 10^{12}$$

$$\epsilon_N = 2 \ \pi\text{-mm-mr}$$

$$\text{B=1 (unbunched beam)}$$

$$\Delta_\nu = -\frac{N r_0}{4\pi\beta\gamma^2\epsilon_N} = -.247$$

This is pretty large, but because this is a rapid cycling machine, it is less sensitive to resonances

Because this affects individual particles, it's referred to as an "incoherent tune shift", which results in a tune spread. There is also a "coherent tune shift", caused by images charges in the walls of the beam pipe and/or magnets, which affects the entire bunch more or less equally.

This is an important effect, but beyond the scope of this lecture.

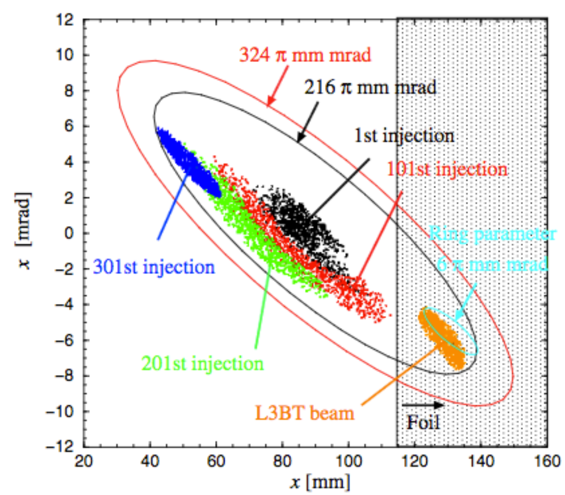
Space Charge Considerations

- In general, space charge tuneshifts limit the total beam that can be injected into a machine. The tuneshift is

$$\Delta\nu = -\frac{NBr_0}{4\pi\beta\gamma^2\epsilon_{x,N}}$$

  and we would like to keep the magnitude below about .2

- One technique is to "paint" the beam to fill the aperture and reduce the normalized emittance
  - Example: The J-PARC in Japan injects 400 MeV beam into their Rapid Cycling Synchrotron and "paints" it to uniformly populate 100 π-mm-mrad



Increasing Energy

- Including different distributions:

"Bunch factor" = I_{peak}/I_{ave}
(Reduce with higher RF harmonics)

total protons

$$\Delta\nu \approx \frac{Nr_0}{2\pi\epsilon_N\beta\gamma^2} FB \lesssim .2$$

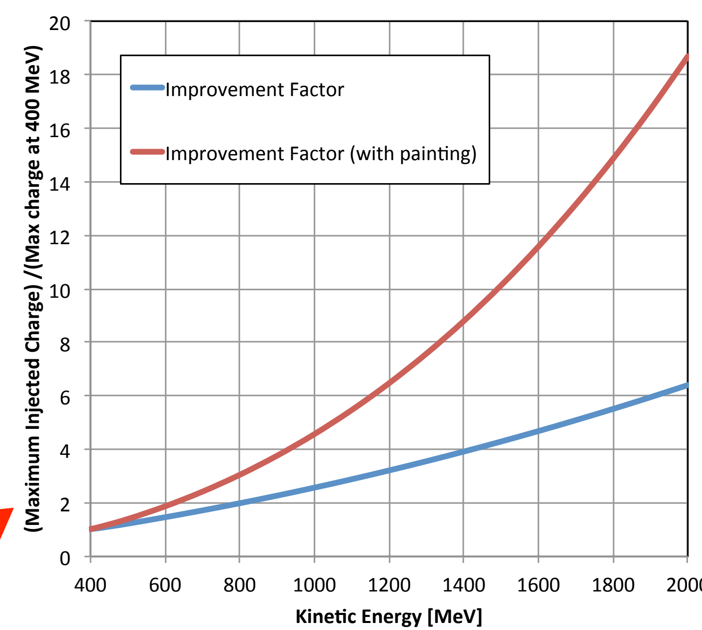
normalized emittance

$\epsilon_N = \epsilon\beta\gamma = \text{constant}$

F = .5 for Gaussian emittance
3 for 95% Gaussian emittance
1 for 100% uniform (painted) emittance

- So the maximum injected charge grows rapidly with increasing energy

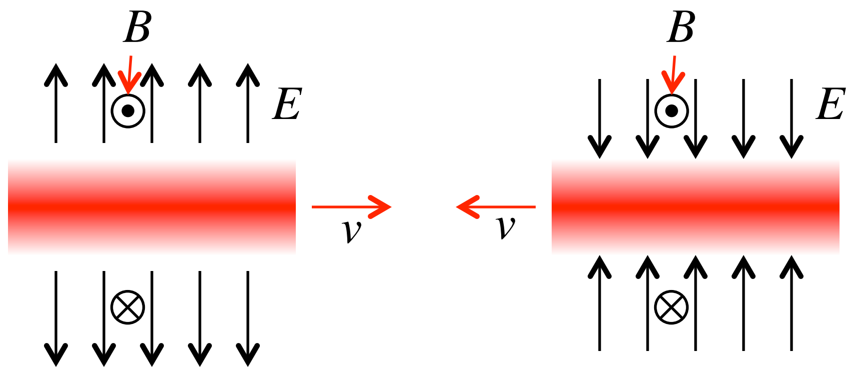
$$N_{max} \propto \beta\gamma^2 \quad \text{without painting}$$
$$\propto \beta^2\gamma^3 \quad \text{painted to fill physical aperture}$$



doesn't include improvement of going to uniform distribution with painting

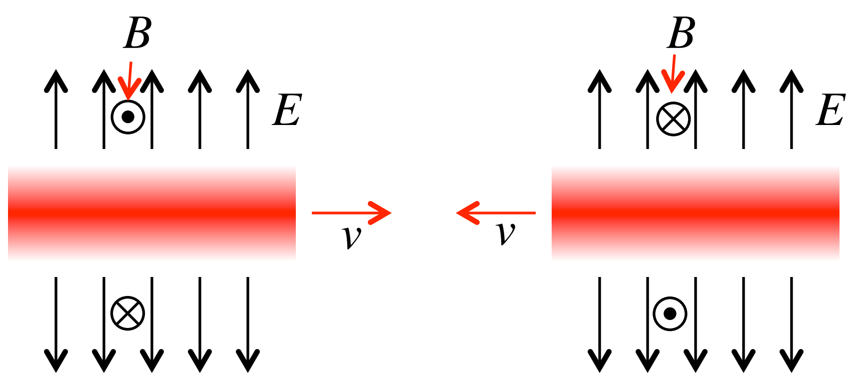
Beam-beam Interaction

If two *oppositely* charged bunches pass through each other…



Both E and B fields are *attractive* to the particles in the other bunch

If two bunches with the *same* sign pass through each other…



Both E and B fields are *repulsive* to the particles in the other bunch

In either case, the forces add

≈ 2

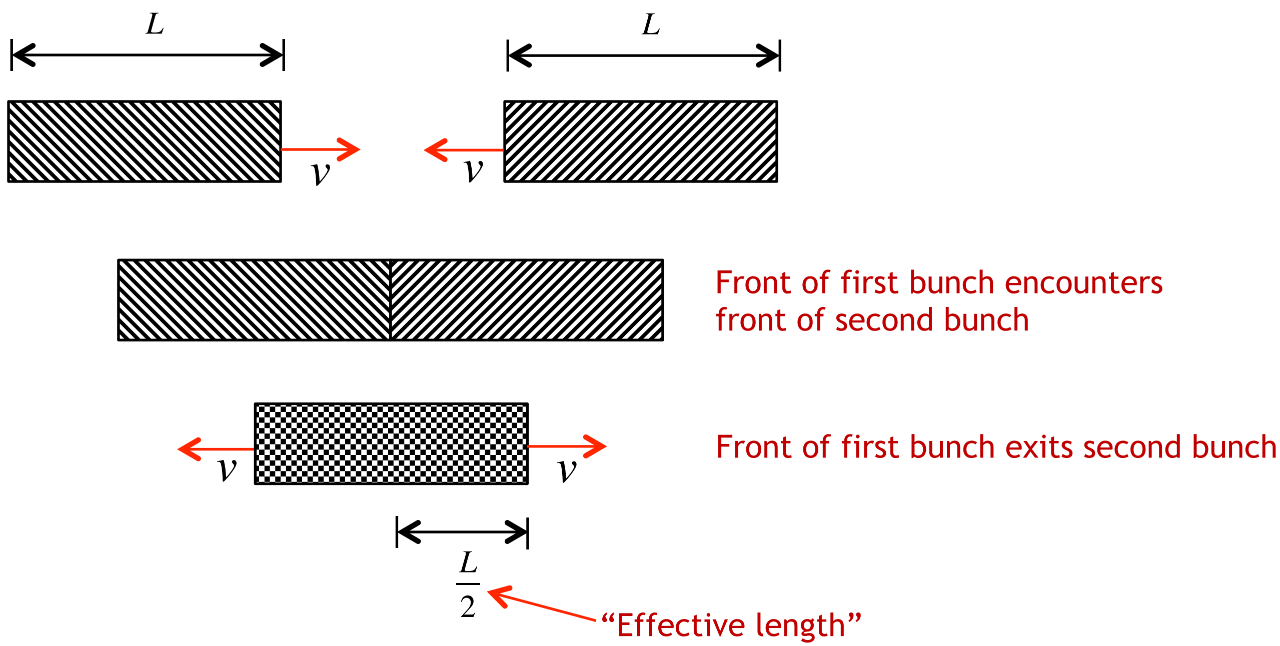
$$\vec{F} = -\hat{r}\frac{e^2}{2\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)\left(1+\beta^2\right)$$

$$\approx -\hat{r}\frac{e^2}{\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$x'' = \frac{F_x}{vp};\quad y'' = \frac{F_y}{vp} \quad \Rightarrow \quad \Delta x' = \frac{F_x}{vp}\Delta s; \quad \Delta y' = \frac{F_y}{vp}\Delta s$$

Integrate…

Effective Length



$$\Delta x' = \frac{F_x}{vp}\Delta s = \frac{F_x}{vp}\left(\frac{L}{2}\right)$$

$$= -\frac{N_b e^2}{2\pi\epsilon_0 r\gamma\beta^2 mc^2}\frac{x}{r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$\beta \approx 1$

$$\approx -\frac{2N_b r_0}{\gamma}\frac{x}{r^2}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$= -\frac{2N_b r_0}{\gamma}\frac{x}{\left(x^2 + y^2\right)}\left(1 - e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

$$\approx -\frac{N_b r_0}{\gamma\sigma^2}x = -\frac{1}{f_{eff}}x$$

Small x and y

$$\Delta y' \approx -\frac{N_b r_0}{\gamma\sigma^2}y = -\frac{1}{f_{eff}}y$$

→ $$\Delta\nu = \frac{N_b r_0 \beta^*}{f_{eff}} = \frac{N_b r_0}{4\pi}\frac{\beta^*}{\gamma\sigma^2}$$

Maximum tuneshift for particles near center of bunch

$$= \frac{N_b r_0}{4\pi\epsilon_N}$$

normalized emittance

$$\equiv \xi$$

"Tuneshift Parameter"

Luminosity and Tuneshift

The total tuneshift will ultimately limit the performance of any collider, by driving the beam onto an unstable resonance. Values of on the order ~.02 are typically the limit. However, we have seen the somewhat surprising result that the tuneshift

$$\xi = \frac{N_b r_0}{2\pi\epsilon\gamma}$$

does not depend on β^*, but only on

$$\frac{N_b}{\epsilon} \equiv \quad \text{"brightness"}$$

For a collider, we have

$$\mathcal{L} = \frac{f n_b N_b^2}{4\pi\sigma^2} = \frac{f n_b N_b^2}{4\pi\left(\frac{\beta^* \epsilon_N}{\gamma}\right)} = \frac{f n_b N_b \gamma}{r_0 \beta^*}\left(\frac{r_0}{4\pi}\frac{N_b}{\epsilon_N}\right)$$

$$= f\frac{n_b N_b \gamma}{r_0 \beta^*}\xi$$

We assume we will run the collider at the “tuneshift limit”, in which case we can increase luminosity by

- Making β^* as small as possible
- Increasing N_b and ε proportionally.