Synchrotron Radiation and Light Sources
For a relativistic particle, the total radiated power (S&E 8.1) is

\[
P = \frac{1}{6\pi\varepsilon_0} \frac{e^2 a^2}{c^3} \gamma^4
\]

\[
\approx \frac{1}{6\pi\varepsilon_0} \frac{e^2 c}{\rho^2} \gamma^4 = \frac{1}{6\pi\varepsilon_0} \frac{e^2 c}{\rho^2} \left( \frac{E}{m_0 c^2} \right)^4
\]

\[a = \text{acceleration} = \frac{v^2}{\rho} \approx \frac{c^2}{\rho}\]

In a magnetic field

\[
\rho = \frac{m\gamma c}{eB} \rightarrow P = \frac{e^4}{6\pi\varepsilon_0} \frac{B^2}{m_0^2 c^2} \gamma^2 = \frac{e^4}{6\pi\varepsilon_0 m^4 c^5} B^2 E^2
\]

Electron radiates \(10^{13}\) times more than a proton of the same energy!
The first attempt to observe synchrotron radiation was in 1944 at the 100 MeV GE betatron.
Because of a miscalculation, they were looking in the microwave region rather than the visible (in fact the walls were opaque), so although they say an energy decay, they did not observe the radiation.

Synchrotron radiation was first successfully observed in 1947 by Elder, Gurewitsch, and Langmuir at the GE 70 MeV electron synchrotron.
Two competing effects

- Damping

\[ \tau_{\Delta E} \approx \tau \frac{E_s}{U_s} \]

\( \tau \) damping time

\( \Delta E \) energy lost per turn

\( E_s \) period

\( U_s \) energy

- Quantum “heating” effects related to the statistics of the photons

\[ N_p = \dot{N} \tau \quad \Rightarrow \quad \sigma_{\Delta E} = \sqrt{\dot{N} \tau_{\Delta E} \langle u^2 \rangle} \]

\( N_p \) Number of photons per period

\( \dot{N} \) Rate of photon emission

\( \sigma_{\Delta E} \) Average photon energy
The power spectrum of radiation is given by

\[
\frac{dP}{d\omega} = \frac{P}{\omega \omega_c} S\left(\frac{\omega}{\omega_c}\right); \quad \omega_c = \frac{3\gamma^3 c}{2 \rho}
\]

\[
S(x) = \frac{9\sqrt{3}}{8\pi} x \int_0^\infty K_{\frac{5}{3}}(u) du
\]

Differential photon rate

\[
\dot{n} = \frac{d\dot{N}}{du}
\]

“critical frequency”

“critical wavelength”

“critical energy”

\[
\lambda_c = \frac{2\pi c}{\omega_c} = \frac{4\pi \rho}{3\gamma^3}
\]

\[
u_c = \frac{3\gamma^3 (\hbar c)}{2 \rho}
\]
The total rate is:

$$\dot{N} = \int_0^\infty \dot{n}(u) du = \frac{15\sqrt{3}}{8} \frac{P}{u_c}$$

The mean photon energy is then

$$\langle u \rangle = \frac{P}{\dot{N}} = \frac{8}{15\sqrt{3}} u_c$$

The mean square of the photon energy is

$$\langle u^2 \rangle = \frac{1}{\dot{N}} \int_0^\infty u^2 \dot{n}(u) du = \frac{P}{\dot{N}} \int_0^\infty \frac{u}{u_c} S \left( \frac{u}{u_c} \right) du$$

$$= \frac{11}{27} u_c^2$$

The energy lost per turn is

$$U_s = \oint P \, dt = \frac{e^2 c \gamma^4}{6\pi\epsilon_0} \oint \frac{1}{\rho^2} \left( \frac{dt}{ds} \right) ds$$

$$= \frac{e^2 \gamma^4}{6\pi\epsilon_0} \oint \frac{1}{\rho^2} ds$$

$$= \frac{1}{c}$$
In 1944 GE looked for synchrotron radiation in a 100 MeV electron beam.

- Assume $B=1$ T

We have

- $E \approx pc = 100$ MeV
- $mc^2 = .511$ MeV
- $\gamma = E/(mc^2) = 196$
- $(B \rho) = 100/300 = .333$ T-m
- $\rho = (B \rho)/B = .333$ m

\[ u_c = \frac{3\gamma^3 (hc)}{2 \rho} = \frac{3(196)^3(1.97 \times 10^{-7})}{2(.333)} = 6.6 \text{ eV} \]

\[ \langle u \rangle = \frac{8}{15\sqrt{3}} u_c = 2.05 \text{ eV} \]

\[ \lambda_{\langle u \rangle} = \frac{hc}{\langle u \rangle} = \frac{1.2}{2.05} = .587 \mu m \]

Visible yellow light, NOT microwaves
It’s important to remember that $\rho$ is not the curvature of the accelerator as a whole, but rather the curvature of individual magnets.

\[ \Delta \theta = \frac{\Delta s}{\rho} \rightarrow \oint \frac{1}{\rho} ds = 2\pi \]

So if an accelerator is built using magnets of a fixed radius $\rho_0$, then the energy lost per turn is

\[ U_s = \frac{e^2 \gamma^4}{6\pi \epsilon_0} \oint \frac{1}{\rho^2} ds = \frac{e^2 \gamma^4}{6\pi \epsilon_0 \rho_0} \oint \frac{1}{\rho} ds = \frac{e^2 \gamma^4}{3\epsilon_0 \rho_0} \]

“isomagnetic”

**Example: CESR**

\[ E = 5.29 \text{ GeV} \]
\[ \rho_0 = 98 \text{ m} \]
\[ U_s = .71 \text{ MeV} \]

\[ \langle u \rangle = \frac{8}{15\sqrt{3}} u_c = .98 \text{ keV} \]

\[ \sqrt{\langle u^2 \rangle} = \sqrt{\frac{11}{27}} u_c = 2.0 \text{ keV} \]

\[ N_s = 721 \text{ photons/turn} \]
Small Amplitude Longitudinal Motion

\[ P \propto E^2 \rightarrow \text{Particles lose more energy at the top of this cycle than the bottom} \]

\[ \Delta E \equiv \varepsilon \]

\[ \theta \]

\[ \Delta t \]

Energy lost and smaller amplitude

Reaccelerate

Smaller amplitude

\[ \int \left( d\varepsilon_0^2 \right) dt = \frac{1}{\tau_s} \int \frac{d\varepsilon_0^2}{dt} dt \]

\[ = -\frac{2}{\tau_s} \int \langle \varepsilon P \rangle dt + \frac{1}{\tau_s} \int \dot{N} \langle u^2 \rangle dt \]

damping term

Heating term due to statistical fluctuations
In general, if I have a simple damping force of the form
\[ \frac{dA}{dt} = -\lambda A \]
the solution is
\[ A(t) = A_0 e^{-\lambda t} = A_0 e^{-t/\tau}; \quad \text{where } \tau = 1 / \lambda \]

If I add a constant heating term
\[ \frac{dA}{dt} = -\lambda A + h \]
then
\[ \int \frac{dA}{A - h / \lambda} = \int -\lambda \, dt \]
\[ \ln(A - h / \lambda) = -\lambda t + K \]
\[ A = Ce^{-\lambda t} + h / \lambda \]
\[ A(0) = A_0 \rightarrow C = 1 + h / \lambda \]
\[ A(t) = A_0 e^{-\lambda t} + \frac{h}{\lambda} \left(1 - e^{-\lambda t}\right) \]
\[ A(\infty) = \frac{h}{\lambda} = h\tau \]
Evaluate integral in damping term

\[ \int \langle \varepsilon P \rangle dt = \frac{1}{c} \int \left( 1 + \frac{x}{\rho} \right) \langle \varepsilon P \rangle ds \]

\[ \approx \frac{1}{c} \int \left( 1 + D \frac{\varepsilon}{\rho E_s} \right) \langle \varepsilon P \rangle ds \]

Recall

\[ P = \frac{e^4}{6 \pi \varepsilon_0 m^4 c^5} B^2 E^2 \]

\[ P(\varepsilon) = P_s \left( 1 + 2 \frac{dB}{B_0 dE} + 2 \frac{1}{E_s} \varepsilon \right) \]

Can’t ignore anything!!

Dependence of field

\[ \frac{dB}{dx} = B' \]

\[ \frac{dB}{dE} = \frac{dB}{dx} \frac{dx}{dE} = \frac{\kappa (B \rho) D}{E_s} \]

\[ P(\varepsilon) = P_s \left( 1 + \frac{2 \varepsilon}{E_s} (\kappa \rho D + 1) \right) \]
Skipping a lot of math...

\[
\left< \frac{d\varepsilon_0^2}{dt} \right> = -\frac{2}{\tau_s} \int \langle \varepsilon P \rangle dt + \frac{1}{\tau_s} \oint \dot{N} \langle u^2 \rangle dt
\]

\[
= \frac{\varepsilon_0^2 U_s}{\tau_s E_s} (2 + D) + \frac{1}{\tau_s} \oint \dot{N} \langle u^2 \rangle dt
\]

\text{damping} \quad \text{heating}

\[
\varepsilon_0^2(t) = \varepsilon_0^2(0) e^{-t/\tau_{\varepsilon^2}} + \varepsilon_0^2(\infty) \left(1 - e^{-t/\tau_{\varepsilon^2}}\right)
\]

where \( \tau_{\varepsilon^2} = \frac{U_s}{\tau_s E_s} (2 + D) \)

\[\text{The energy then decays in a time}\]

\[
\tau_{\varepsilon} = 2\tau_{\varepsilon^2}
\]

\[
\frac{1}{\tau_{\varepsilon}} = \frac{U_s}{2\tau_s E_s} (2 + D)
\]
So far we have talked about “separated function”, “isomagnetic” lattices, which has

- A single type of dipole: \( \kappa = 0; \rho = \rho_0 \)
- Quadrupoles: \( \kappa \neq 0; \rho = \infty \)

In this case

\[
D \equiv \frac{1}{\rho^2} \left( 2\kappa \rho D + \frac{D}{\rho} \right) ds = \frac{1}{\rho_0^2} \frac{D}{\rho_0} ds = \frac{1}{\rho_0^2} (C\alpha_C)
\]

\[
= \frac{C\alpha_C}{2\pi \rho_0} \approx \alpha_C \ll 1
\]

\[
\frac{1}{\tau_E} \approx \frac{U_s}{\tau_s E_s}
\]

probably the answer you would have guessed without doing any calculations.
We can relate the spread in energy to the peak of the square with

$$\sigma_{\varepsilon}^2 = \left\langle \varepsilon_0^2(\infty) \right\rangle = \frac{1}{2} \varepsilon_0^2(\infty)$$

$$= \frac{1}{2} \frac{\tau_{\varepsilon}^2}{\tau_s} \int \left\langle \dot{\varepsilon}u^2 \right\rangle dt = \frac{\tau_{\varepsilon}}{4\tau_s} \int \left\langle \dot{\varepsilon}u^2 \right\rangle dt = \frac{E_s}{2U_s(2 + D)} \int \left\langle \dot{\varepsilon}u^2 \right\rangle dt$$

Use

$$P = \frac{1}{6\pi\varepsilon_0} \frac{e^2c}{\rho^2} \gamma^4, \dot{N} = \frac{15\sqrt{3}}{8} \frac{P}{u_c}, \left\langle u^2 \right\rangle = \frac{11}{27} u_c^2, \quad u_c = \frac{3}{2} \frac{\hbar\gamma^3}{\rho}$$

$$\tau_{\varepsilon} = \tau_s \frac{2E_s}{U_s(2 + D)}, U_s = \frac{e^2\gamma^4}{3\varepsilon_0\rho_0}$$
This leads to

\[
\int \langle \dot{N} u^2 \rangle \, dt = \frac{55}{16\sqrt{3}} \frac{e^2 \hbar c \gamma^7}{6\pi \varepsilon_0} \int \frac{1}{\rho^3} \, ds
\]

\[
= \frac{55}{16\sqrt{3}} \frac{e^2 (\hbar c) \gamma^7}{3\varepsilon_0 \rho_0^2}
\]

\[
\sigma_{\varepsilon}^2 = \frac{E_s}{2U_s (2 + D)} \left( \frac{55}{16\sqrt{3}} \frac{e^2 (\hbar c) \gamma^7}{3\varepsilon_0 \rho_0^2} \right)
\]

\[
= \frac{E_s}{(1 + D)} \frac{55}{32\sqrt{3}} \frac{(\hbar c) \gamma^3}{\rho_0}
\]

\[
= \frac{E_s}{(2 + D)} \frac{55}{32\sqrt{3}} \frac{\hbar (\gamma mc^2)}{mc \rho_0} \gamma^2
\]

\[
= C_q \frac{\gamma^2 E_s^2}{(2 + D) \rho_0}
\]

\[
C_q \equiv \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.8 \times 10^{-13} \text{ m} \quad \text{(for electrons)}
\]
Synchrotron radiation

Energy lost along trajectory, so radiated power will reduce momentum along flight path

\[ \frac{d\rho}{dt} \approx -\frac{P}{c} \hat{\theta} \]

If we assume that the RF system restores the energy lost each turn, then

Energy lost along the path \( \rightarrow \Delta y = \Delta y' = 0 \)
Energy restored along nominal path \( S \) \( \Rightarrow \) ”adiabatic damping”
Skipping a lot of math, the damping time in the vertical plane is given by

\[
\frac{1}{\tau_y} = \frac{1}{2\tau_s} \frac{U_S}{E_S} = \frac{1}{2\tau_\varepsilon}
\]

Note, here are no heating terms in the vertical plane! This is why electron machines have “flat” beams. In the absence of any perturbations, the vertical emittance will damp to ~0, which could cause stability problems.
The horizontal plane has the same damping term as the vertical plane, but it has more contributions because the position depends on energy.

$$\Delta E$$

$$x = x_\beta + D \frac{\epsilon}{E_s}$$

$$x' = x'_\beta + D' \frac{\epsilon}{E_s}$$

$$x_\beta = a\sqrt{\beta} \cos(\psi(s) + \delta) \equiv a\sqrt{b}C$$

$$x'_\beta = -\frac{a}{\sqrt{\beta}} \left( \alpha \cos(\psi(s) + \delta) + \sin(\psi(s) + \delta) \right) \equiv -\frac{a}{\sqrt{\beta}} (\alpha C + S)$$

If we radiate a photon of energy $$u$$, it will change the energy, but not the position or the angle.

$$\Delta x = \left[ (x_\beta + \Delta x_\beta) + D \frac{\epsilon - u}{E_s} \right] - \left[ x_\beta + D \frac{\epsilon}{E_s} \right]$$

$$= \Delta x_\beta - D \frac{u}{E_s} = 0$$

$$\Delta x_\beta = D \frac{u}{E_s}$$

$$\Delta x' = \Delta x'_\beta - D' \frac{u}{E_s} = 0$$

$$\Delta x'_\beta = D' \frac{u}{E_s}$$
Again, skipping a lot of math, we get

\[
\frac{1}{\tau_x} \approx \frac{U_s}{2\tau_s E_s} (1 - D)
\]

where

\[
D = \frac{\int \frac{1}{\rho^2} \left( 2\kappa \rho D + \frac{D}{\rho} \right) ds}{\int \frac{1}{\rho^2} ds}
\]

Same as longitudinal plane

Separated function
isomagnetic synchrotrons
Note:

\[
\frac{1}{\tau_e} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{U_s}{2\tau_s E_s} (2 + D)
\]

\[
+ \frac{U_s}{2\tau_s E_s} (1 - D)
\]

\[
+ \frac{U_s}{2\tau_s E_s}
\]

\[
= \frac{2U_s}{\tau_s E_s}
\]

This is called Robinson's theorem and it's always true. For a separated function, isomagnetic lattice, it simplifies to

\[
\frac{1}{\tau_e} = \frac{1}{\tau_x} = \frac{1}{\tau_y} = \frac{U_s}{\tau_s E_s}
\]
The equilibrium emittance is given by

\[ \epsilon_x(\infty) = C_q \frac{\gamma^2}{(1 - D)} \oint \frac{\mathcal{H}}{\rho^3} ds \]

where \( C_q \equiv \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.8 \times 10^{-13} \text{m} \) (for electrons)

\[ \mathcal{H} = \gamma D^2 + 2\alpha DD' + \beta D'^2 \]

For a separated function, isomagnetic machine, this becomes

\[ \epsilon_x(\infty) = C_q \frac{\gamma^2}{2\pi \rho_0 (1 - D)} \oint \frac{\mathcal{H}}{\rho} ds \]

With some handwaving, this can be approximated by

\[ \epsilon_x(\infty) \approx C_q \gamma^2 \frac{R}{\rho_0} \frac{1}{\nu_x^3} \]
For a separated function, isomagnetic synchrotron

Energy lost per turn \( U_s = \frac{e^2 \gamma^4}{3\varepsilon_0 \rho_0} \); for electrons \( U_s [\text{MeV}] = 0.0885 \frac{E^4 [\text{GeV}]}{\rho_0 [\text{m}]} \)

Longitudinal damping time \( \tau_\varepsilon \approx \tau_s \frac{E_s}{U_s} \)

Transverse damping times
\[ \tau_x \approx 2\tau_s \frac{E_s}{U_s} \]
\[ \tau_y \approx \tau_x \]
\[ \frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{2U_s}{\tau_s E_s} \] — Robinson’s Theorem (always true)

Equilibrium energy spread \( \sigma_\varepsilon^2(\infty) \approx C_q \frac{\gamma^2 E_s^2}{2\rho_0} \); for electrons \( C_q \equiv \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.8 \times 10^{-13} \text{ m} \)

Equilibrium horizontal emittance \( \epsilon_x(\infty) \approx C_q \gamma^2 \frac{R}{\rho_0 \nu_x^3} \)
Benefits of Damping

- Can inject off orbit and beam will damp down to equilibrium
  - Don’t have to worry about painting or charge exchange like protons.
  - Can inject over many turns, or even continuously.
- Beams will naturally “cool” (i.e. reduce their emittance in phase space)
- Example: Beams injected off orbit into CESR

![Graph showing beam cooling over time](image1)

![Graph showing intensity vs. energy difference](image2)
Considerations for $e^+e^-$ Colliders

- In the case of proton-proton and proton-antiproton colliders, we assumed:
  - The optics were the same in the two planes
  - The emittances were the same in the two planes
  - The normalized emittance was preserved.

- This allowed us to write

\[
L = f \frac{N_b^2}{4\pi \sigma_x^2} = f_{\text{rev}} \frac{1}{4\pi} n_b N_b^2 \frac{\gamma}{\beta^*_x \epsilon_x}
\]

- In general, none of this will be true for $e^+e^-$ colliders:
  - The emittance will be much smaller in the $y$ plane
  - Because the emittance is large in the $x$ plane, we will not be able to “squeeze” the optics as far without hitting the aperture in the focusing triplet, so in general, $\beta^*_x > \beta^*_y$.

- We must write

\[
L = f \frac{N_1 N_2}{4\pi \sigma_x \sigma_y} = f_{\text{rev}} \frac{1}{4\pi} n_b \frac{N_1 N_2}{\sqrt{\beta^*_x \epsilon_x \beta^*_y \epsilon_y}}
\]

Unnormalized(!) emittance
Shortly after the discovery of synchrotron radiation, it was realized that the intense light that was produced could be used for many things:

- Radiography
- Crystallography
- Protein dynamics
- ...

The first “light sources” were parasitic on electron machines that were primarily used for other things.

As the demand grew, dedicated light sources began to emerge.

The figure of merit is the “brightness”:

\[
\text{photons/s/mm}^2 / \text{mrad}^2 / (\text{bandwidth})
\]
These just used the parasitic synchrotron light produced by the bend dipoles.

Examples

- **SURF** (1961): 180 MeV UV synchrotron at NBS
- **CESR** (CHESS, 70’s): 6 GeV synchrotron at Cornell
- Numerous others

Typically large emittances, which limited brightness of the beam.
Examples:

- 1981: 2 GeV SRS at Daresbury (\(\varepsilon=106 \text{ nm-rad}\))
- 1982: 800 MeV BESSY in Berlin (\(\varepsilon=38 \text{ nm-rad}\))
- 1990: SPEAR II becomes dedicated light source (\(\varepsilon=160 \text{ nm-rad}\))

Often include “wigglers” to enhance SR
Typical 2nd Generation Parameters

- **Beam sizes**
  - $\sigma_y \sim 1$ mm
  - $\sigma_y' \sim 0.1$ mrad
  - $\sigma_x \sim 1$ mm
  - $\sigma_x' \sim 0.03$ mrad

- **Broad spectrum**

- **High flux**
  - Typically $10^{13}$ photons/second/mrad for 3 GeV, 100 mA dipole source at $E_{\text{crit}}$
In rest frame of electron \( \lambda^* = \frac{\lambda_U}{\gamma} \)

Electron oscillates coherently with (contracted) structure, and releases photons with the same wavelength.

In the lab frame, this is Doppler shifted, so

\[
\lambda = \frac{\lambda^*}{2\gamma} = \frac{\lambda_U}{2\gamma^2}
\]

So, \( \lambda \) on the order of 1cm \( \Rightarrow \) X-rays.
Bends, Undulators, and Wigglers*

Bend — Undulator — Wiggler

- Electron (e−) propagation

- Flux \( \propto \sqrt{\hbar \omega} \)
- Brightness \( \propto \hbar \omega \)
- Flux \( \propto \hbar \omega \)

- White source
- Partially coherent source
- Powerful white source

*G. Krafft
3rd Generation (Undulator) Sources

- High Brightness
  - $10^{19}$ compared to $10^{16}$ for 2nd generation sources
  - Emittance ~1-20 nm-rad

- A few Examples:
  - CLS
  - SPEAR-III
  - Soleil
  - Diamond
  - APS
  - PF
  - NSLS
  - BESSY
  - Doris
  - ...
Fourth Generation light sources generally utilize free electron lasers (FELs) to increase brightness by at least an order of magnitude over Third Generation light sources by using coherent production (see Bryant Garcia’s talk).
LCLS-II at SLAC

- 4 GeV superconducting linac
- 1 MHz operation
- X-rays up to 25 keV
Evolution of Parameters
Light Sources are a Huge (and growing) Industry

- Wikipedia lists about 60 light sources worldwide