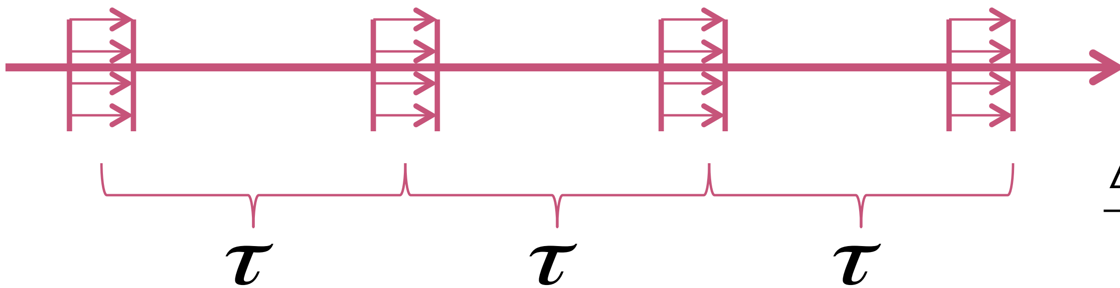
Longitudinal Motion

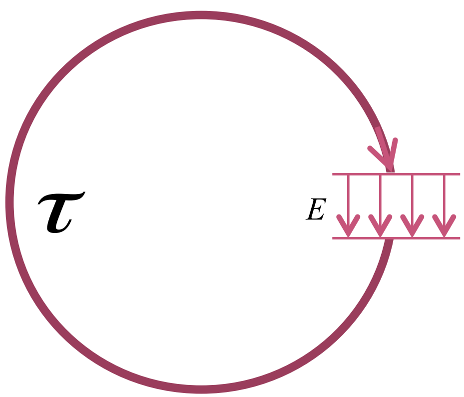
Acceleration in Periodic Structures

- We consider motion of particles either through a linear structure or in a circular ring



Always negative

$$\frac{\Delta\tau}{\tau} = -\frac{1}{\gamma^2}\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$



In both cases, we can adjust the RF phases such that a particle of nominal energy arrives at the the same point in the cycle φ_s

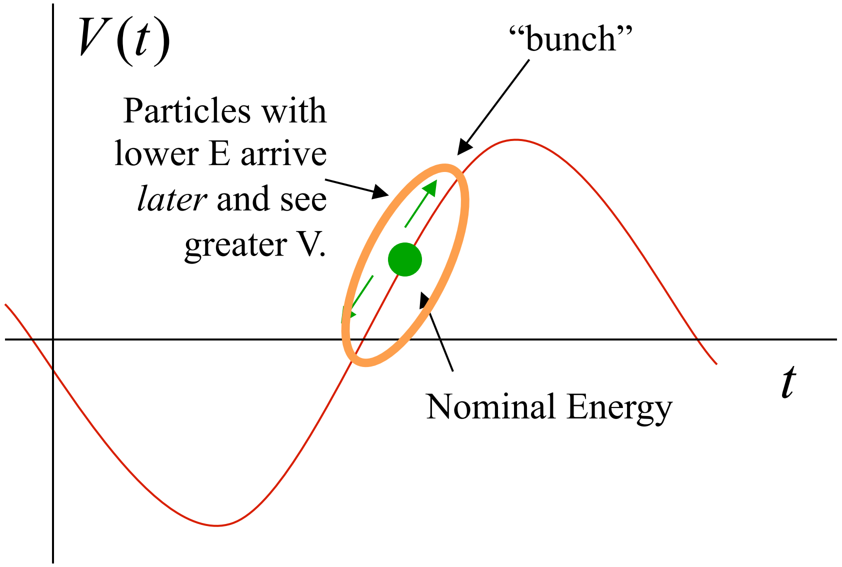
$$\frac{\Delta\tau}{\tau} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$

Goes from negative to positive at transition

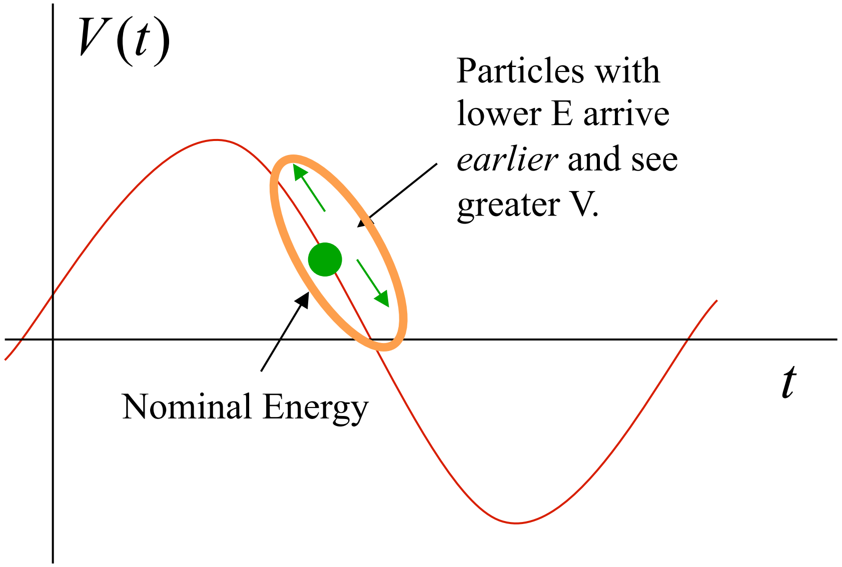
Slip Factors and Phase Stability

- The sign of the slip factor determines the stable region on the RF curve.

η<0 (linacs and below transition)

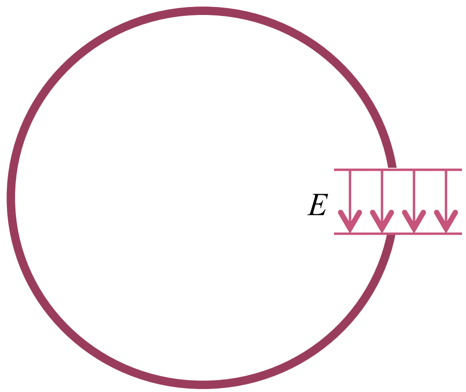


η>0 (above transition)



Longitudinal Acceleration

- Consider a particle circulating around a ring, which passes through a resonant accelerating structure each turn



$$V = V_0 \sin(\omega_{RF} t + \omega_0); \quad \omega_{RF} = 2\pi \frac{h}{\tau}$$

Harmonic number (number of RF cycles per period) → h

Period of nominal energy particle → τ

- The energy gain that a particle of the nominal energy experiences each turn is given by

$$E_{n+1} = E_n + eV_0 \sin\phi_s$$

← Synchronous phase

where this phase will be the same for a particle on each turn

- A particle with a different energy will have a different phase, which will evolve each turn as

$$\phi_{n+1} = \phi_n + \Delta\phi$$

← Phase difference from one turn to the next

Phase Difference

- The phase difference can be expressed as

$$\Delta\phi = \omega_{RF}\Delta t = \omega_{RF}\tau\eta\frac{\Delta p}{p} = 2\pi h\frac{1}{\beta^2}\frac{\Delta E}{E}$$

use $\dfrac{\Delta p}{p} = \dfrac{1}{\beta^2}\dfrac{\Delta E}{E}$

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{E_s\beta^2}\Delta E$$

"synchronous" (nominal) energy

Longitudinal Equation of Motion

- Thus the change in energy for this particle for this particle will evolve as

$$\Delta E_{n+1} = \Delta E_n + eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

- So we can write

Constant (or very slowly varying)

$$\frac{d\phi}{dn} = \frac{2\pi h\eta}{E_S\beta^2}\Delta E$$

$$\frac{d\Delta E}{dn} = eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

$$\Rightarrow \frac{d^2\phi}{dn^2} = \frac{eV_0 2\pi h\eta}{E_S\beta^2}\left(\sin\phi_n - \sin\phi_s\right)$$

exact

Synchrotron Motion and Synchrotron Tune

- Rewrite this equation as:

$$\frac{d^2\phi}{dn^2}+\left(-\frac{eV_0 2\pi h\eta}{E_S\beta^2}(\sin\phi_n-\sin\phi_s)\right)=0$$

- For *small* oscillations,

$$\sin\phi_n-\sin\phi_s\approx\left.\frac{d\sin\phi}{d\phi}\right|_{\phi=\phi_s}(\phi_n-\phi_s)=\Delta\phi\cos\phi_s$$

- And we have

$$\frac{d^2\Delta\phi}{dn^2}+\left(-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s\right)\Delta\phi=0$$

- This is the equation of a harmonic oscillator with

$$\omega_n=\sqrt{-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s}\Rightarrow\nu_s=\frac{1}{2\pi}\sqrt{-\frac{eV_0 2\pi h\eta}{E_S\beta^2}\cos\phi_s}$$

Angular frequency wrt *turn* (not time)

"synchrotron tune" = number of oscillations per turn (usually <<1)

Longitudinal Phase Space and Emittance

- We want to write things in terms of time and energy. We have can write the longitudinal equations of motion as

$$\Delta t(n) = \frac{1}{\omega_{rf}} \Delta\phi(n)$$

$$\frac{d\Delta t(n)}{dn} = \frac{1}{\omega_{rf}} \frac{d\Delta\phi(n)}{dn} = \frac{\tau\eta}{E_S\beta^2} \Delta E(n)$$

- Following our procedure for longitudinal motion, we want to write this in form:

$$\Delta t(n) = a\cos(2\pi\nu_s n) + b\sin(2\pi\nu_s n)$$

$$\Delta t(0) \equiv \Delta t_0 = a \rightarrow a = \Delta t_0$$

$$\left.\frac{d\Delta t(n)}{dn}\right|_{n=0} = 2\pi\nu_s b = \frac{\tau\eta}{E_S\beta^2}\Delta E_0 \rightarrow b = \frac{\tau\eta}{2\pi E_S\beta^2\nu_s}$$

$$\Delta t(n) = \Delta t_0 \cos\left(2\pi\nu_s n\right) + \frac{\tau\eta}{2\pi E_S\beta^2\nu_s}\Delta E_0 \sin\left(2\pi\nu_s n\right)$$

- Taking the derivative wrt *n* and substituting for Δ*E* gives us

$$\Delta E(n) = \frac{E_S\beta^2}{\tau\eta}\frac{d\Delta t(n)}{dn} = \Delta E_0 \cos\left(2\pi\nu_s n\right) - \frac{2\pi E_S\beta^2\nu_s}{\tau\eta}\Delta t_0 \sin\left(2\pi\nu_s n\right)$$

- So we can write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_s n) & \dfrac{\tau\eta}{2\pi E_S\beta^2\nu_s}\sin(2\pi\nu_s n) \\ -\dfrac{2\pi E_S\beta^2\nu_s}{\tau\eta}\sin(2\pi\nu_s n) & \cos(2\pi\nu_s n) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- Wait! We've seen this before. This looks just like our equation for longitudinal motion, but with α=0, so we immediately write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_s n) & \beta_L \sin(2\pi\nu_s n) \\ -\gamma_L \sin(2\pi\nu_s n) & \cos(2\pi\nu_s n) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- Where

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \frac{\tau|\eta|}{2\pi E_S\beta^2\nu_s} = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}};\gamma_L = \frac{1}{\beta_L}$$

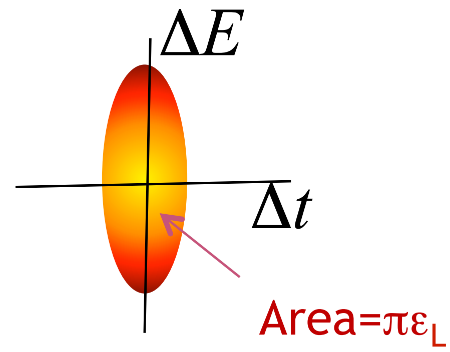
Units: s/eV

Units: eV/s

- We can define an invariant of the motion as

$$\frac{1}{\beta_L}(\Delta t)^2 + \beta_L(\Delta E)^2 \equiv \epsilon_L = \text{constant}$$

units
generally eV-s



- What about the behavior of *Δt* and *ΔE* separately?

$$\Delta E_{RMS} = \sqrt{\frac{\epsilon_L}{\beta_L}} = \left(-\frac{\epsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta}\right)^{\frac{1}{4}}$$

$$\Delta t_{RMS} = \sqrt{\epsilon_L \beta_L} = \left(-\frac{\epsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}\right)^{\frac{1}{4}}$$

$$\Rightarrow \frac{\Delta E}{\Delta t} = \beta_L$$

Aspect ratio

- Note that for linacs or well-below transition

$$\eta = -\frac{1}{\gamma^2} \Rightarrow \Delta E \propto \left(\gamma^3\beta^2\right)^{\frac{1}{4}}; \Delta t \propto \left(\gamma^3\beta^2\right)^{-\frac{1}{4}}$$

Large Amplitude Motion

- We have solved for the synchrotron tune in the limit of small oscillations, but in general we will not restrict ourselves to small oscillations.
- Recall our exact equations of motion:

$$\frac{d\phi}{dn}=\frac{2\pi h\eta}{E_s\beta^2}\Delta E$$

substitute

$$\frac{d\Delta E}{dn}=eV_0\left(\sin\phi_n-\sin\phi_s\right)$$

$$\Rightarrow\frac{d^2\phi}{dn^2}=\frac{eV_0 2\pi h\eta}{E_s\beta^2}\left(\sin\phi_n-\sin\phi_s\right)$$

- Multiply both sides by $\frac{d\phi}{dn}$ and integrate over *dn*

constant

$$\int\left(\frac{d\phi}{dn}\frac{d^2\phi}{dn^2}\right)dn=\frac{eV_0 2\pi h\eta}{E_s\beta^2}\int\left(\sin\phi(n)-\sin\phi_s\right)\frac{d\phi}{dn}dn$$

$$\Rightarrow\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2=-\frac{eV_0 2\pi h\eta}{E_s\beta^2}\left(\cos\phi+\phi\sin\phi_s\right)+\text{constant}$$

$$\Rightarrow\boxed{\left(\Delta E\right)^2+2\frac{eV_0E_s\beta^2}{2\pi h\eta}\left(\cos\phi+\phi\sin\phi_s\right)=\text{constant}}$$

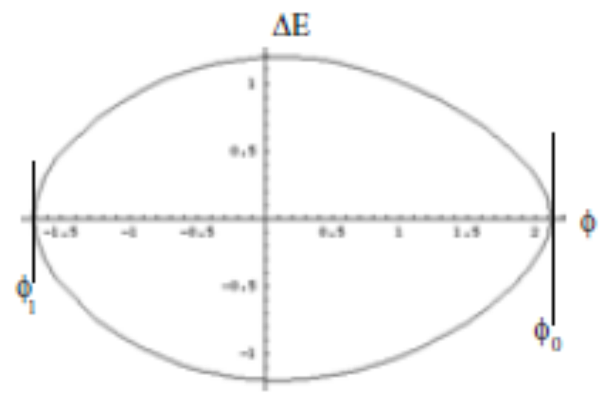
exact

- After a lot of messy (and not very intuitive) math, this becomes…

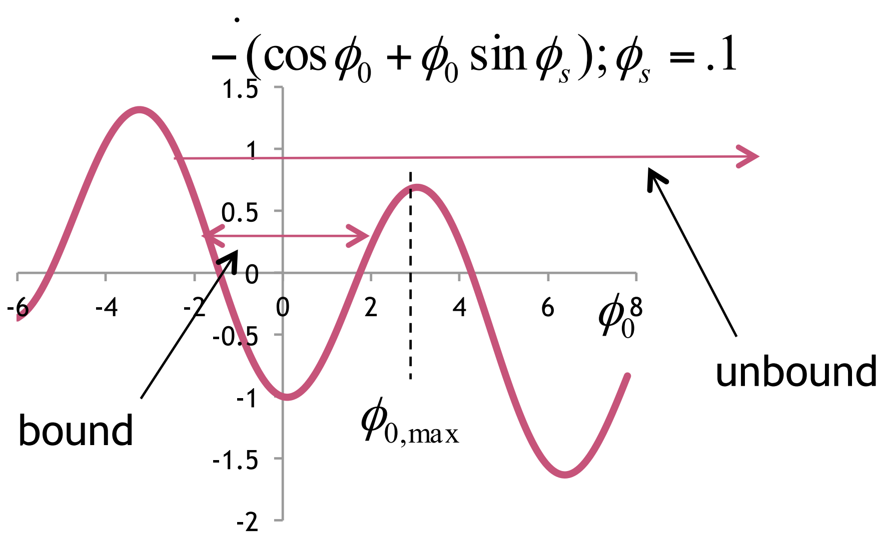
$$\frac{1}{2}(\Delta E)^2 = \frac{\left[\cos\phi + \phi\sin\phi_s\right] - \left[\cos\phi_0 + \phi_0\sin\phi_s\right]}{\omega_{rf}^2\cos\phi_s\beta_L^2}$$

- The curve will cross the ϕ axis when $\Delta E=0$, which happens at two points defined by

$$\cos\phi_1 + \phi_1\sin\phi_s = \cos\phi_0 + \phi_0\sin\phi_s$$



- Phase trajectories are possible up to a maximum value of ϕ_0. Consider



Limit is at maximum of

$$\sin\phi_{0,\max} - \sin\phi_s = 0$$

$$\Rightarrow \phi_{0,\max} = \phi_s \text{ or } \boxed{\pi - \phi_s}$$

Longitudinal Separatrix

- The other bound of motion can be found by

$$\cos\phi_{1,\max} + \phi_{1,\max}\sin\phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s$$
$$= -\cos\phi_s + (\pi - \phi_s)\sin\phi_s$$

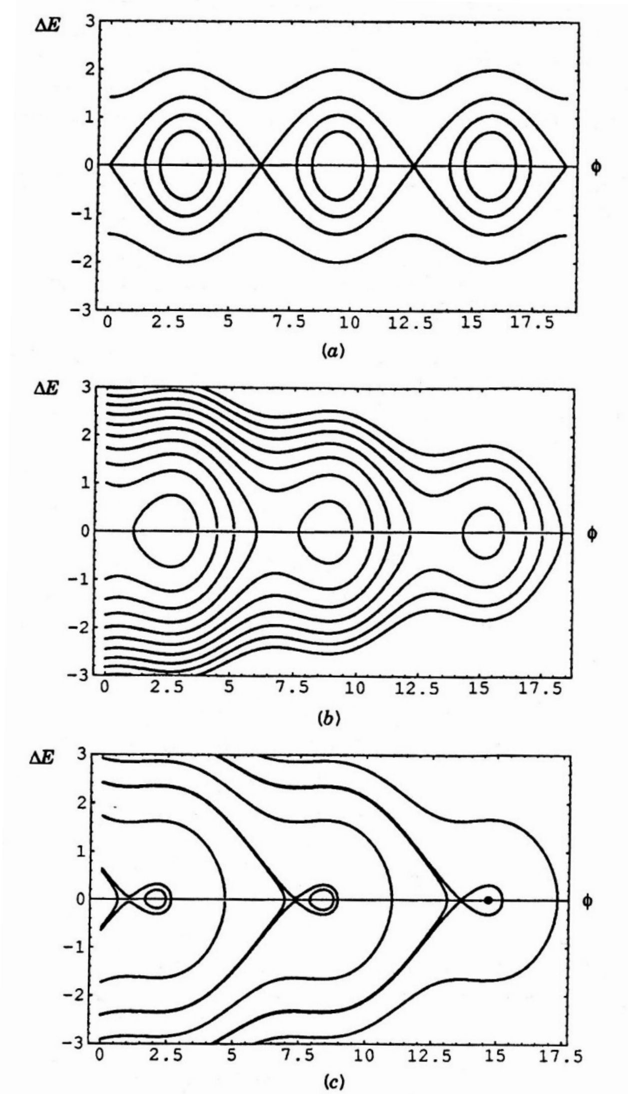
- The limiting boundary (separatrix) is defined by

$$(\Delta E)^2 = 2\frac{\left[\cos\phi + \phi\sin\phi_s\right] + \left[\cos\phi_s - (\pi - \phi_s)\sin\phi_s\right]}{\omega_{rf}^2\cos\phi_s\beta_L^2}$$

- The maximum energy of the "bucket" can be found by setting $\phi=\phi_s$

$$(\Delta E_b)^2 = 2\frac{2\cos\phi_s + 2\phi_s\sin\phi_s - \pi\sin\phi_s}{\omega_{rf}^2\cos\phi_s\beta_L^2}$$

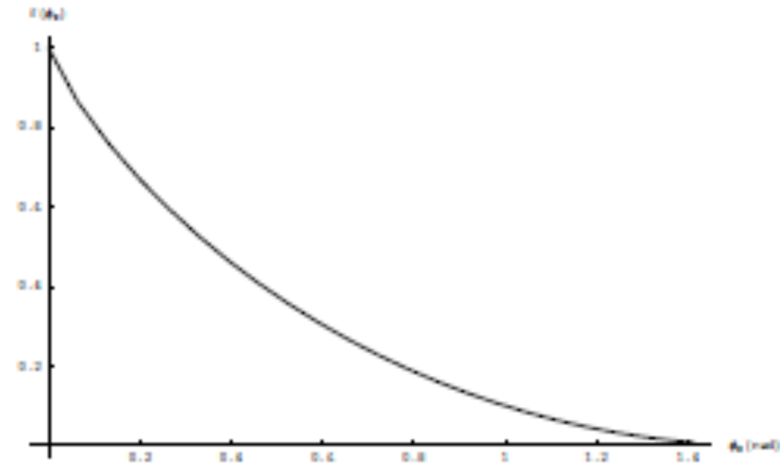
$$\Delta E_b = 2\frac{\sqrt{1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s}}{\omega_{rf}\beta_L}$$



Bucket Area

- The bucket area can be found by integrating over the area inside the separatrix (which I won't do)

$$A_b = \frac{16\beta}{\omega_{rf}}\sqrt{\frac{eV \cdot E_s}{2\pi h|\eta|}} f(\phi_s); \quad f(\phi_s) =$$



Transition Crossing

- We learned that for a simple FODO lattice $\gamma_T \approx \nu$ so electron machines are always above transition.
- Proton machines are often designed to accelerate through transition.
- As we go through transition $(\eta < 0) \Rightarrow (\eta = 0) \Rightarrow (\eta > 0)$
- Recall

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}} = \frac{\Delta t_{max}}{\Delta E_{max}}$$

At transition:

$\Delta t_{max} \Rightarrow \text{constant}$

$\Delta E_{max} \Rightarrow \infty$

so these both go to zero at transition.

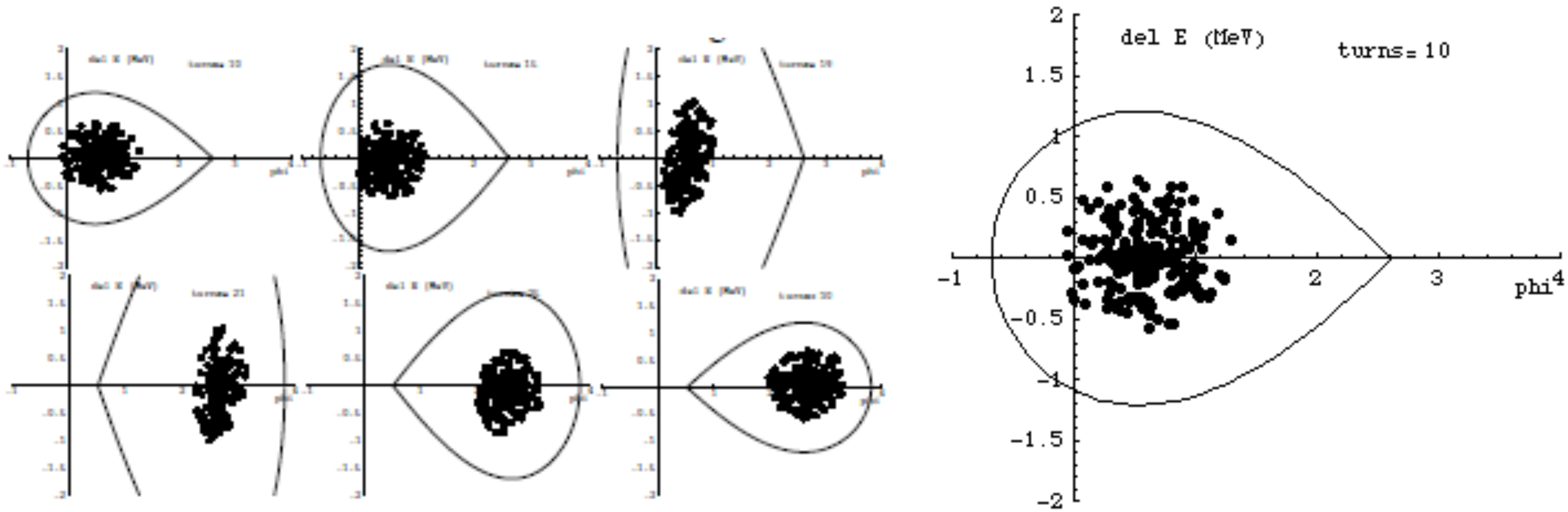
- To keep motion stable

$$\cos\phi_s > 0 \text{ below transition;} \Rightarrow 0 < \phi_s < \frac{\pi}{2}$$

$$\cos\phi_s < 0 \text{ above transition;} \Rightarrow \frac{\pi}{2} < \phi_s < \pi$$

Effects at Transition

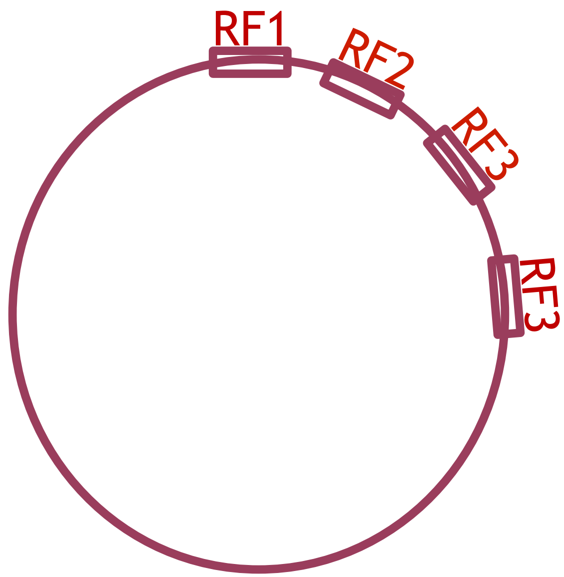
- As the beam goes through transition, the stable phase must change



- Problems at transition (pretty thorough treatment in S&E 2.2.3)
  - Beam loss at high dispersion points
  - Emittance growth due to non-linear effects
  - Increased sensitivity to instablities
  - Complicated RF manipulations near transition
    - Much harder before digital electronics

RF Manipulations

- As you'll show in homework, the synchrotron tune (longitudinal oscillations/turn) is generally <<1.
- That means that if there are multiple RF cavities around the ring, the orbiting particle will see the *vector sum* of the cavities.



$$\frac{\Delta E}{dn} = \sum_{i=1}^{N} V_i \sin(\phi_i)$$
$$= V_{eff} \sin\left(\phi_{eff}\right)$$

ϕ_i is the phase angle at the arrival of the particle at cavity *i*

- We will clearly get the maximum energy gain if all phases are the same, so (assuming all voltages are the same)

$$\frac{\Delta E}{dn} = NV_0 \sin\left(\phi_s\right)$$

Do we always want the maximum acceleration?

- As we'll see, there are times when we want to change the amplitude of the RF quickly.
- Because cavities represent stored energy, changing their amplitude quickly can be difficult.
- Much quicker to change phase
- Standard technique is to divide RF cavities into two groups and adjust the relative phase. In the simplest case, we put half the RF cavities into group "A" and half into group "B". We can adjust the phases of these cavities relative to our nominal synchronous phase as

$$V_{eff}\sin(\phi_{eff}) = \frac{N}{2}V_0\sin(\phi_s+\delta) + \frac{N}{2}V_0\sin(\phi_s-\delta)$$
$$= \frac{N}{2}(\sin\phi_s\cos\delta + \cos\phi_s\sin\delta + \sin\phi_s\cos\delta - \cos\phi_s\sin\delta)$$
$$= NV_0\cos\delta\sin\phi_s$$
$$V_{eff} = NV_0\cos\delta;\phi_{eff} = \phi_s$$

- So $\delta = 0 \Rightarrow V_{eff} = NV_0$

$$\delta = \frac{\pi}{2} \Rightarrow V_{eff} = 0$$

Like "turning RF off"

Adiabatic Capture

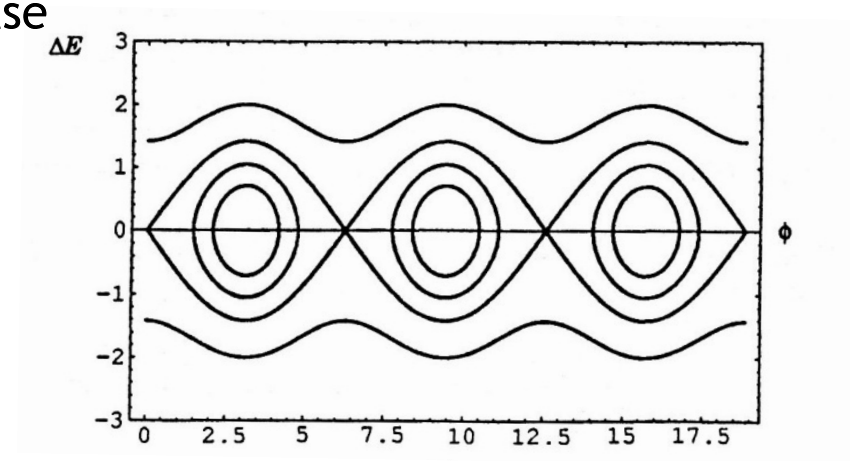
- We initially capture the beam by raising the RF voltage "adiabatically" (over many synchrotron oscillations). This insures that the longitudinal phase space stays matched to the RF bucket

- If instead we simply turn the RF on, the beam will "filament".

- (simulations: IOTA ring, protons, K=2.5 MeV, τ=1.77 μsec, h=4, V_{max}=1kV)

Capture and Acceleration

- We can adiabatically capture beam by increasing the RF voltage with no accelerating phase



- As we accelerate beam, Δt decreases. Recall $\frac{1}{\beta_L}(\Delta t)^2 + \beta_L(\Delta E)^2 \equiv \varepsilon_L = \text{constant}$

$$\Delta E_{RMS} = \sqrt{\frac{\varepsilon_L}{\beta_L}} = \left(-\frac{\varepsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta}\right)^{\frac{1}{4}}$$

$$\Delta t_{RMS} = \sqrt{\varepsilon_L \beta_L} = \left(-\frac{\varepsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}\right)^{\frac{1}{4}}$$

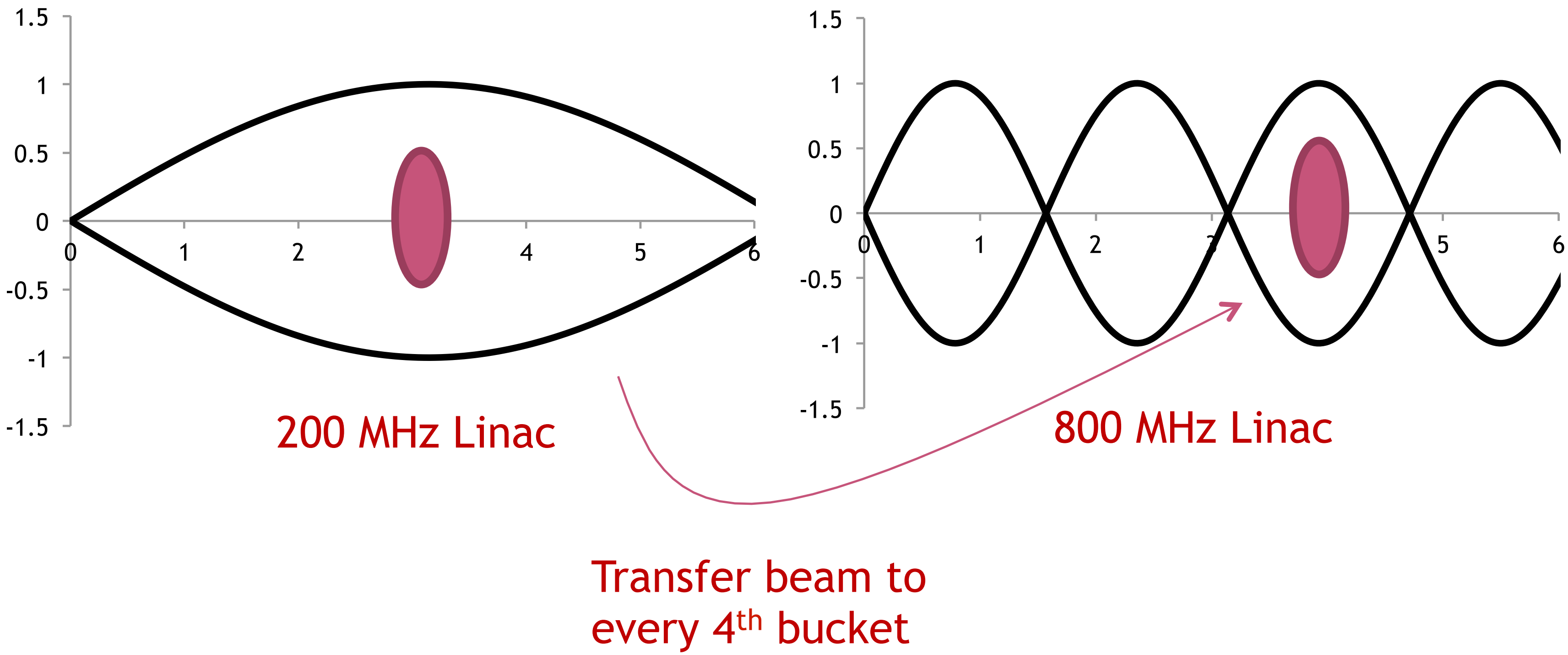
- So as beam accelerates, bunches get narrower

Bucket to Bucket Transfer

- In general, the accelerating gradient of an RF structure is

$$\frac{V}{L} \propto \frac{V_{breakdown}}{\lambda_{RF}} \propto \omega_{RF} V_{breakdown}$$

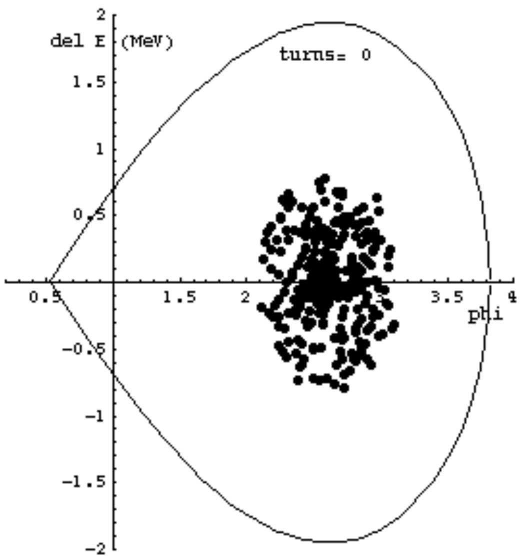
- So when bunches get short enough, it's advantageous to transfer to a higher frequency. For example, in the Fermilab Linac



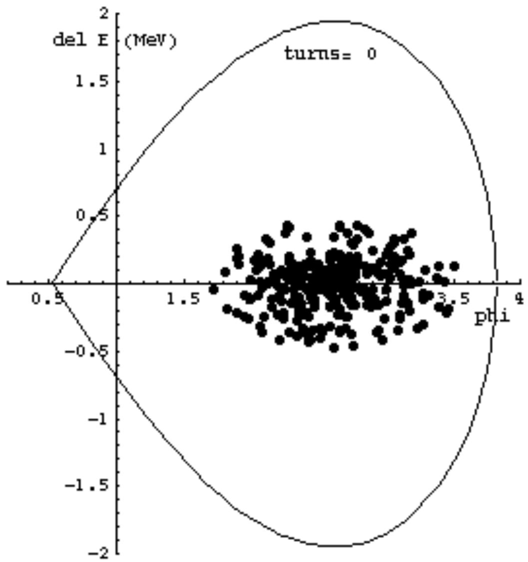
Bucket Transfers: Phase and Beta Matching

- When we transfer beam from one machine to another, or from a lower frequency section to a higher frequency section, it's important to correctly match the phases!

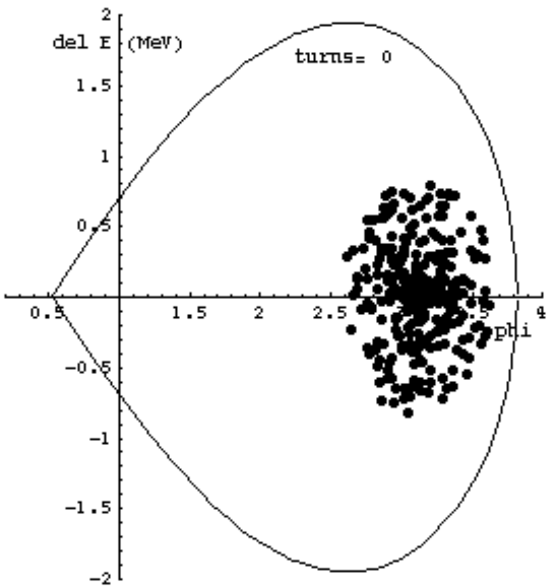
Matched transfer



β_L mismatch



Phase mismatch



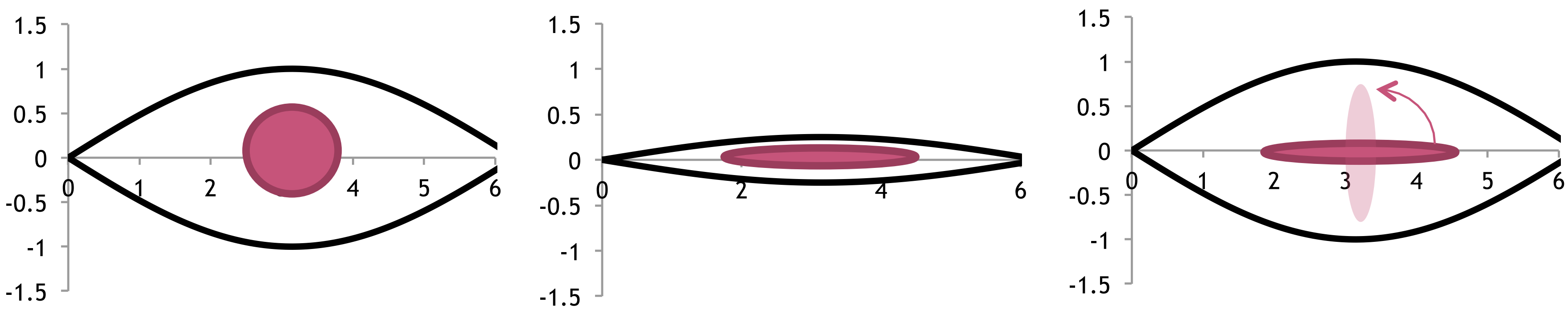
Bunch Rotation

- If we *slowly* change the RF voltage (or effective voltage by phasing), we can adiabatically change the bunch shape

$$\Delta E_{RMS} = \sqrt{\frac{\varepsilon_L}{\beta_L}} = \left(-\frac{\varepsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s}{\tau\eta} \right)^{\frac{1}{4}} \propto V_0^{\frac{1}{4}}$$

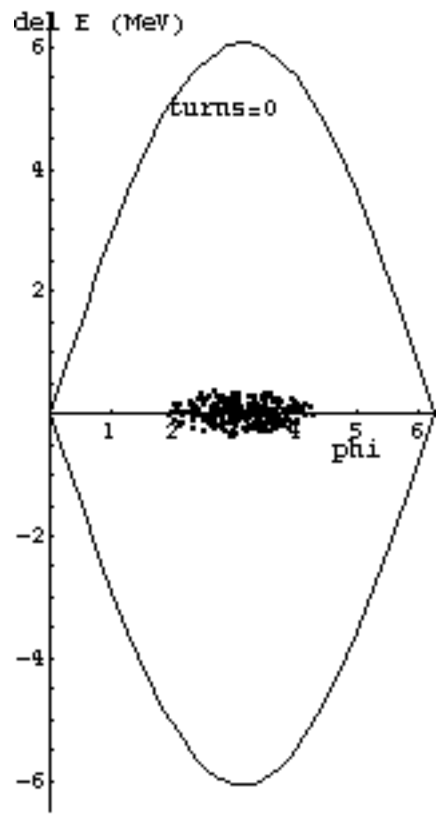
$$\Delta t_{RMS} = \sqrt{\varepsilon_L \beta_L} = \left(-\frac{\varepsilon_L^2 \tau\eta}{e V_0 \omega_{rf} E_S \beta^2 \cos\phi_s} \right)^{\frac{1}{4}} \propto V_0^{-\frac{1}{4}}$$

- If we suddenly change the voltage, then the bunch will be mismatched and will rotate in longitudinal phase space



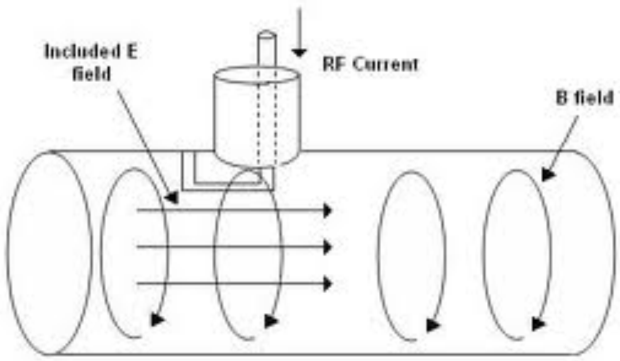
Bunch Rotation (cont'd)

- Of course, non-adiabatically increasing the RF voltage ("snapping") will cause the beam to filament, but the effect is minor over ¼ of a synchrotron oscillation



Accelerating Structures

- The basic resonant structure is the "pillbox"



$$\vec{E} = E(r,t)\hat{z}$$

$$\vec{B} = B(r,t)\hat{\phi}$$

In cylindrical coords, Maxwell's Equations Become:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial E_z}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial E_z}{\partial r} = \frac{\partial B_\theta}{\partial t}$$

Boundary Conditions:

$$E_{\|} = B_{\perp} = 0$$

Differentiating the first by δt and the second by δr:

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right) = \frac{\partial B_\theta}{\partial r \partial t} + \frac{1}{r}\frac{\partial B_\theta}{\partial t} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial r^2} = \frac{\partial B_\theta}{\partial r \partial t}$$

$$\Rightarrow \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

- General solution of the form

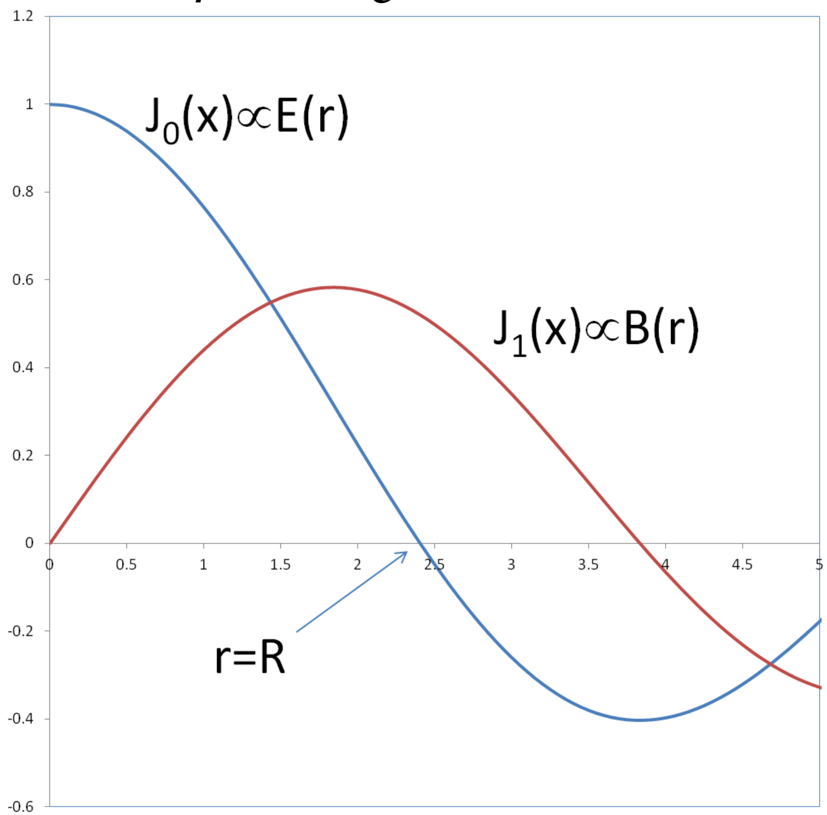
$$E_z = E(r)e^{+i\omega t}$$

- Which gives us the equation

0th order Bessel's Equation

$$E_z'' + \frac{1}{r}E_z' + \frac{\omega^2}{c^2}E_z = 0 \Rightarrow E(r) = E_0 J_0\left(\frac{\omega}{c}r\right)$$

0th order Bessel function



First zero at *J(2.405)*, so lowest mode

$$f_0 = 2.405\frac{c}{2\pi R}$$

$$\frac{\partial E_z}{\partial r} = E_0\frac{\omega}{c}J_0'\left(\frac{\omega}{c}r\right)e^{i\omega t} = \frac{\partial B_\theta}{\partial t} = i\omega B(r)e^{i\omega t}$$

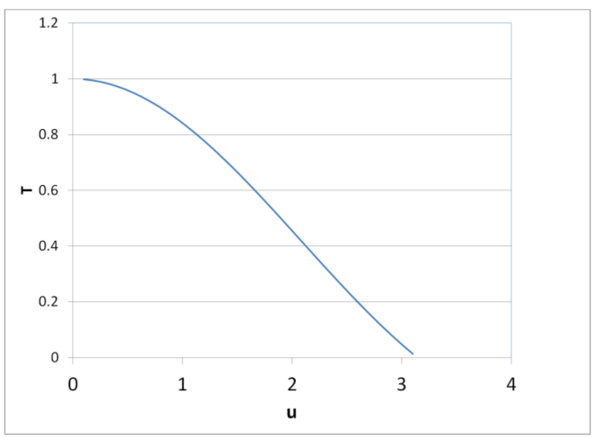
$$\Rightarrow B(r) = -i\frac{1}{c}E_0 J_0'\left(\frac{\omega}{c}r\right) = i\frac{1}{c}E_0 J_1\left(\frac{\omega}{c}r\right)$$

Transit Factor

- In the lowest pillbox mode, the field is uniform along the length ($v_p=\infty$), so it will be changing with time as the particle is transiting, thus a very long pillbox would have no net acceleration at all. We calculate a "transit factor"

Assume peak in middle

$$T \equiv \frac{\text{energy gain}}{eE_0L} = \frac{eE_0\int_{-L/2}^{L/2}\cos\left(2\pi f\frac{z}{v}\right)dz}{eE_0L} = \frac{\frac{v}{\pi f}eE_0\sin\left(\frac{\pi f L}{v}\right)}{eE_0L} = \frac{\sin\left(\frac{\pi f L}{v}\right)}{\left(\frac{\pi f L}{v}\right)} \equiv \frac{\sin u}{u}$$



Example:

- 5 MeV Protons ($v\sim.1c$)
- f=200MHz
- T=85%➔$u\sim1$

$$R = 2.405\frac{c}{2\pi f} = 57\text{ cm}$$

$$L = u\frac{v}{\pi f} = 4.9\text{ cm}$$

Sounds kind of short, but is that an issue?

Power dissipation in RF Cavities

- Energy stored in cavity

Volume=$L\pi R^2$

$$U=\frac{1}{2}\int\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)dV=\frac{1}{2}\varepsilon_0\int E_{max}dV=\frac{1}{2}\varepsilon_0LE_0^2\int_0^R 2\pi rJ_0^2\left(\frac{2\pi f}{c}r\right)dr=\frac{1}{2}\varepsilon_0VE_0^2J_1^2(2.405)$$

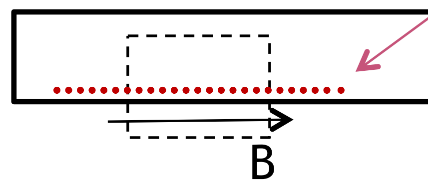
$=(.52)^2\sim25\%$

- Power loss:

Magnetic field at boundary

$$\oint\vec{B}\bullet d\vec{l}=LB_\theta=\mu_0I_{enclosed}=\mu_0LJ$$

$$\Rightarrow J=\frac{1}{\mu_0}B_\theta$$



Surface current density J [A/m]

Average power loss per unit area is

$$\langle p\rangle=\langle\rho_sJ^2\rangle=\frac{1}{2}\frac{1}{\mu_0^2}\rho_s|B_\theta|^2$$

Average over cycle

Cylinder surface — 2 ends

$$P=\frac{1}{2}\rho_s\left(\frac{E_0}{\mu_0c}\right)^2\left(2\pi RLJ_1^2\left(\frac{\omega}{c}R\right)+2\times2\pi\int_0^R J_1^2\left(\frac{\omega}{c}r\right)rdr\right)$$

$$=\frac{1}{2}\rho_s\left(\frac{E_0}{Z_0}\right)^2 2\pi RL\left(1+\frac{R}{L}\right)J_1^2(2.405)$$

where $Z_0\equiv\sqrt{\frac{\mu_0}{\varepsilon_0}}=\mu_0c=\frac{1}{\varepsilon_0c}=376.73\,\Omega$ (impedance of free space)

- The figure of merit for cavities is the Q, where

$$Q \equiv 2\pi \frac{(\text{Stored Energy})}{(\text{Energy Lost per Cycle})} = \omega \frac{U}{P}$$

$$= \omega \frac{\varepsilon_0 E_0^2 \pi R^2 L J_1^2(2.405)}{\rho_s \left(\frac{E_0}{Z_0}\right)^2 2\pi R L \left(1 + \frac{R}{L}\right) J_1^2(2.405)}$$

$$= \frac{Z_0^2 \omega R \varepsilon_0}{2\rho_s \left(1 + \frac{R}{L}\right)} = 2.405 \frac{Z_0^2 c \varepsilon_0}{2\rho_s \left(1 + \frac{R}{L}\right)}$$

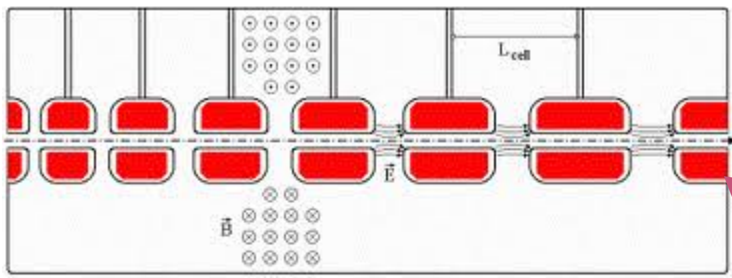
$$c\varepsilon_0 = \frac{1}{Z_0}$$

$$= 2.405 \frac{Z_0}{2\rho_s \left(1 + \frac{R}{L}\right)}$$

- So Q not very good for short, fat cavities!

Drift Tube (Alvarez) Cavity

- Put conducting tubes in a larger pillbox, such that inside the tubes *E=0*



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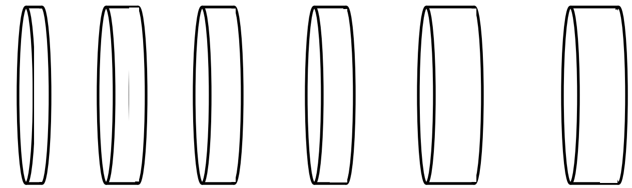
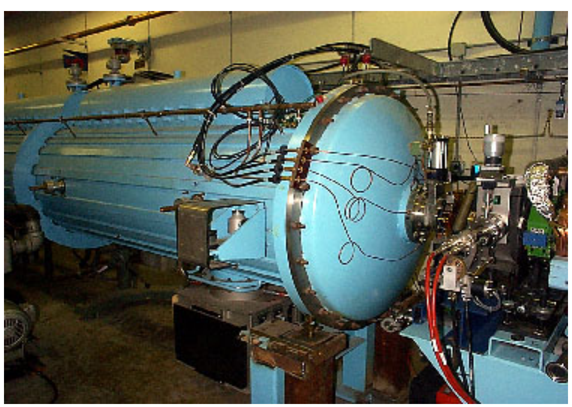
Bunch of pillboxes



$$d = \frac{v}{f}$$

Gap spacing changes as velocity increases

Drift tubes contain quadrupoles to keep beam focused



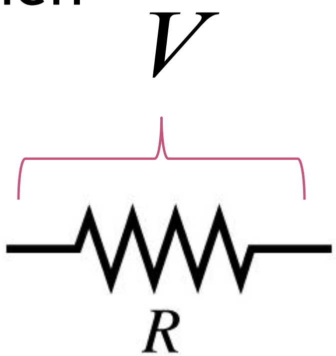
Fermilab low energy linac

→

Inside

Shunt Impedance

- If we think of a cavity as resistor in an electric circuit, then



$$P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P}$$

- By analogy, we define the "shunt impedance" for a cavity as

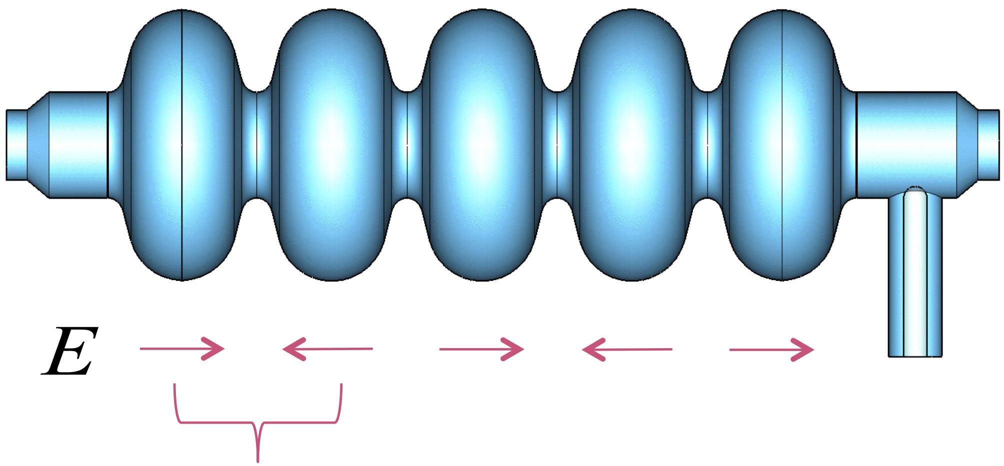
$$R_s \equiv \frac{(\text{voltage gain})^2}{P} = \frac{(E_0 L T)^2}{P}$$

$$= \frac{Z_0^2}{\pi \rho_s} \frac{L}{R} \frac{T^2}{\left(1 + \frac{R}{L}\right) J_1^2(2.405)}$$

We want R_s to be as large as possible

Other Types of Accelerating Structures

- π cavities

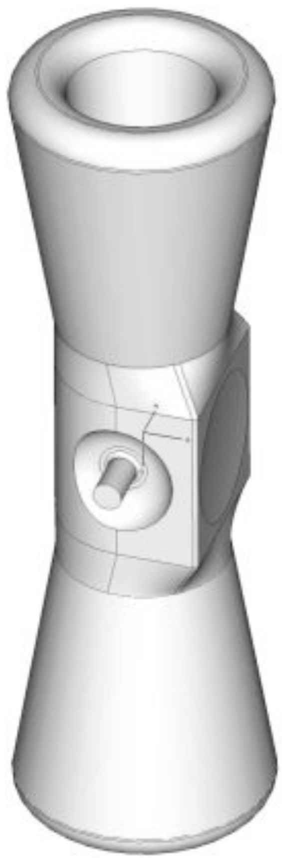


$$d = \pi \frac{v}{\omega}$$

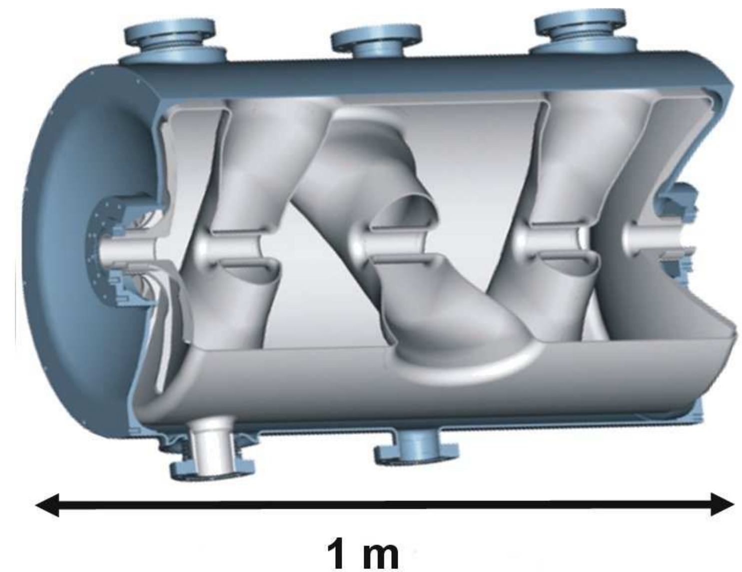
- These are phased so the particles always see acceleration.
- Not practical when particle velocity is changing quickly
  - For higher energy particles, p cavities are "beta-matched" by section.

RF Cavities (cont'd)

- Half-Wave Resonators and Spoke Resonators are optimized for low-β acceleration.



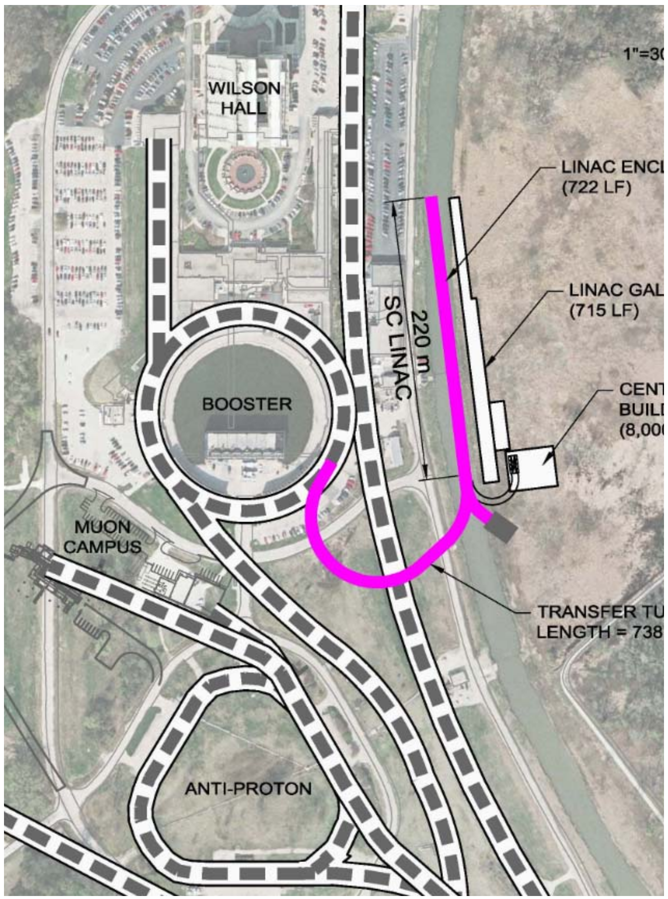
Half-wave Resonator

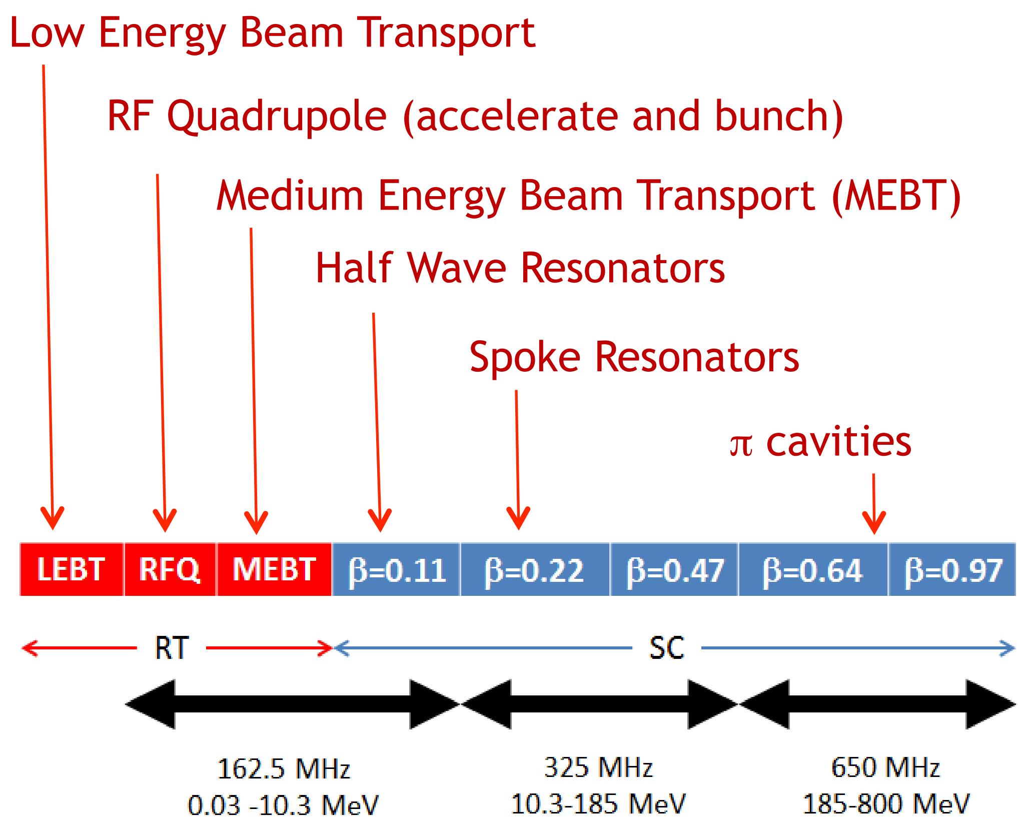


(Triple) Spoke Resonator

Example: Proposed PIP-II Front End

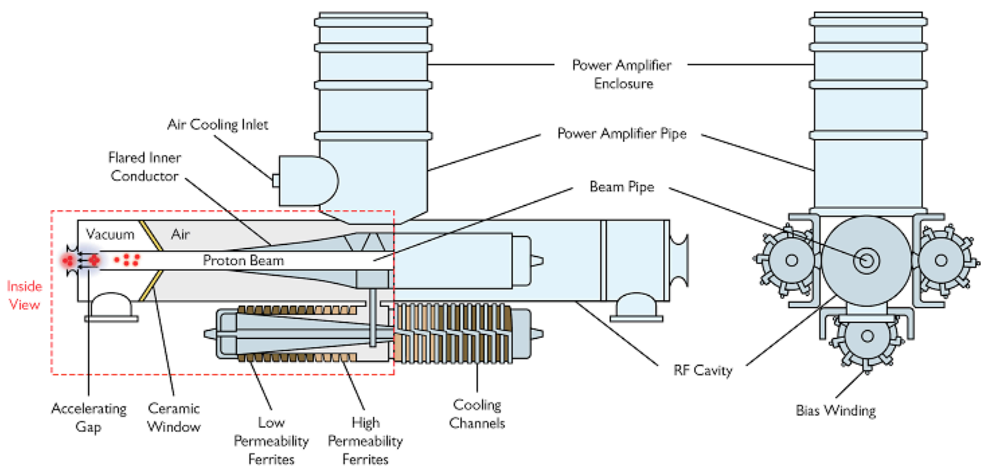
- PIP-II is the linac which is proposed to replace the existing Fermilab Linac

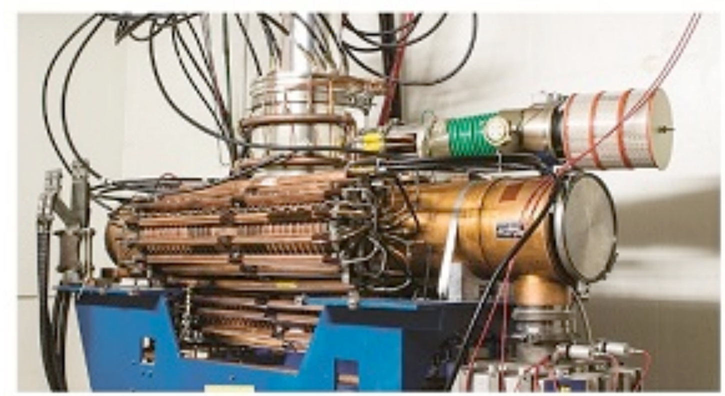




RF Cavities in Rings

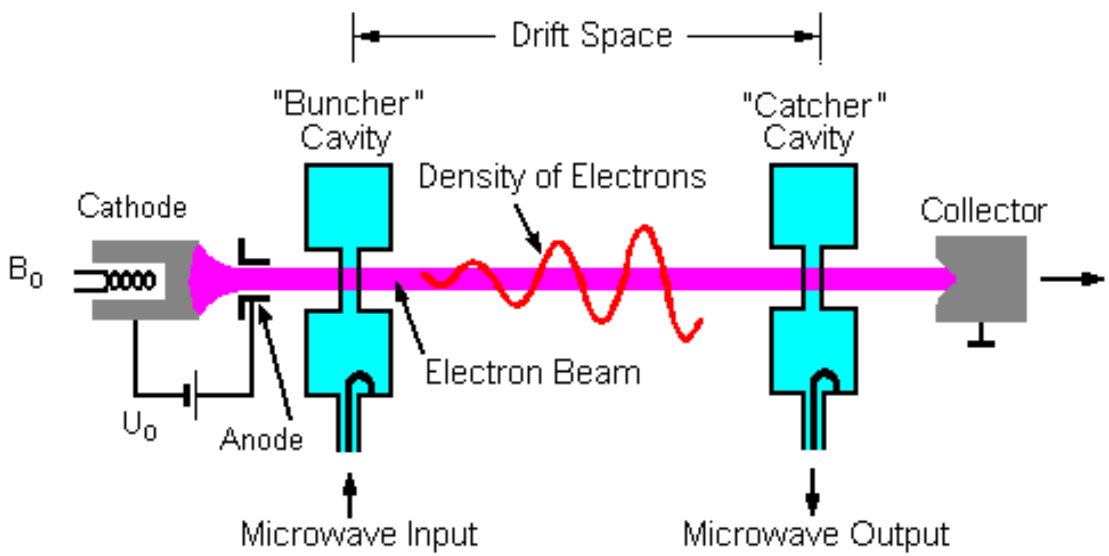
- RF Cavities in rings typically do not have to have high accelerating gradients, but they may have other challenges, particularly at low energy.
- Example: Fermilab Booster cavities must change resonant frequency from 37 to 53 MHz, as the beam goes from 400 MeV to 8 GeV
  - Biased ferrite "tuners" use magnetic fields to load ferrite into its saturation region and thereby change its inducance.





Sources of RF Power

- For frequencies above ~300 MHz, the most common power source is the “klystron”, which is actually a little accelerator itself



- Electrons are bunched and accelerated, then their kinetic energy is extracted as microwave power.

Sources of RF Power (cont'd)

- For lower frequencies (<300 MHz), the only sources significant power are triode tubes, which haven't changed much in decades.

FNAL linac 200 MHz Power Amplifier

53 MHz Power Amplifier for Booster RF cavity