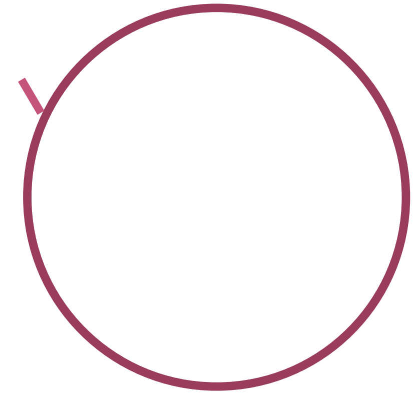
Lattice Imperfections and Off-momentum Particles

Lattice Imperfections

- Up until now, we have considered an ideal lattice, but real magnets aren’t perfect.
- We will consider two types of lattice imperfections:
  - Dipole errors
  - Quadrupole errors
- We will also discuss how to locally correct the position of the beam.

Closed Orbit Distortion ("cusp")



- We place a dipole at one point in a ring which bends the beam by an amount Θ.
- The new equilibrium orbit will be defined by a trajectory which goes once around the ring, through the dipole, and then returns to its exact initial conditions. That is

$$\mathbf{M}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \Rightarrow (\mathbf{I}-\mathbf{M})\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (\mathbf{I}-\mathbf{M})^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

- Recall that we can express the transfer matrix for a complete revolution as

$$\mathbf{M}(s+C,s) = \begin{pmatrix} \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu & \beta(s)\sin 2\pi\nu \\ -\gamma(s)\sin 2\pi\nu & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu \end{pmatrix} = \mathbf{I}\cos 2\pi\nu + \mathbf{J}\sin 2\pi\nu = e^{\mathbf{J}2\pi\nu}$$

$$(\mathbf{I}-\mathbf{M}) = e^{\mathbf{J}\pi\nu}\left(e^{-\mathbf{J}\pi\nu} - e^{\mathbf{J}\pi\nu}\right) = -e^{\mathbf{J}\pi\nu}\left(2\sin\pi\nu\mathbf{J}\right)$$

$$(\mathbf{I}-\mathbf{M})^{-1} = \left(-2\sin\pi\nu\mathbf{J}\right)^{-1}\left(e^{\mathbf{J}\pi\nu}\right)^{-1}$$

$$= \frac{1}{2\sin\pi\nu}\mathbf{J}e^{-\mathbf{J}\pi\nu} = \frac{1}{2\sin\pi\nu}\mathbf{J}\left(\mathbf{I}\cos\pi\nu - \mathbf{J}\sin\pi\nu\right)$$

$$= \frac{1}{2\sin\pi\nu}\left(\mathbf{J}\cos\pi\nu + \mathbf{I}\sin\pi\nu\right)$$

$$= \frac{1}{2\sin\pi\nu}\begin{pmatrix} \alpha\cos\pi\nu + \sin\pi\nu & \beta\cos\pi\nu \\ -\gamma\cos\pi\nu & -\alpha\cos\pi\nu + \sin\pi\nu \end{pmatrix}$$

$$\mathbf{J} \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\mathbf{J}^2 = -\mathbf{I}$$

$$\mathbf{J}^{-1} = -\mathbf{J}$$

- Plug this back in $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{1}{2\sin\pi\nu}\begin{pmatrix} \alpha\cos\pi\nu + \sin\pi\nu & \beta\cos\pi\nu \\ -\gamma\cos\pi\nu & -\alpha\cos\pi\nu + \sin\pi\nu \end{pmatrix}\begin{pmatrix} 0 \\ \theta \end{pmatrix}$

$$= \frac{\theta}{2\sin\pi\nu}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

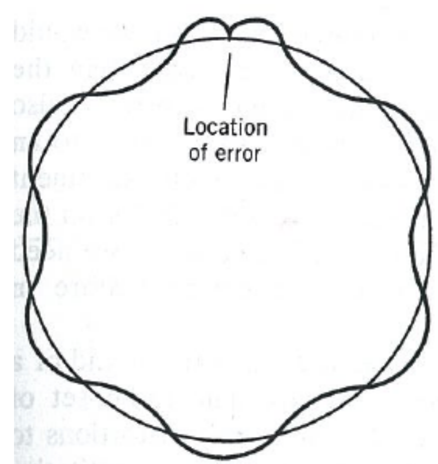
- We now propagate this around the ring

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \frac{\theta}{2\sin\pi\nu}\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\psi + \alpha_0\sin\Delta\psi\right) & \sqrt{\beta_0\beta(s)}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta(s)}}\left((\alpha_0 - \alpha(s))\cos\Delta\psi - (1+\alpha_0\alpha(s))\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\psi - \alpha(s)\sin\Delta\psi\right) \end{pmatrix}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$\Rightarrow x(s) = \frac{\theta}{2\sin\pi\nu}\left(\sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\psi + \alpha_0\sin\Delta\psi\right)\beta_0\cos\pi\nu + \sqrt{\beta_0\beta(s)}\sin\Delta\psi\left(\sin\pi\nu - \alpha_0\cos\pi\nu\right)\right)$$

$$= \frac{\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\left(\cos\Delta\psi\cos\pi\nu + \sin\Delta\psi\cos\pi\nu\right)$$

$$= \frac{\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos\left(\Delta\psi - \pi\nu\right)$$



Quadrupole Errors

- We can express the matrix for a complete revolution at a point as

$$\mathbf{M}(s) = \begin{pmatrix} \cos 2\pi\nu + \alpha(s)\sin 2\pi\nu & \beta(s)\sin 2\pi\nu \\ -\gamma(s)\sin 2\pi\nu & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu \end{pmatrix}$$

- If we add focusing quad at this point, we have

$$\mathbf{M}'(s) = \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 & \cos 2\pi\nu - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\pi\nu_0 + \alpha(s)\sin 2\pi\nu_0 - \frac{\beta(s)}{f}\sin 2\pi\nu_0 & \beta(s)\sin 2\pi\nu_0 \\ -\gamma(s)\sin 2\pi\nu_0 - \frac{1}{f}\left(\cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0\right) & \cos 2\pi\nu_0 - \alpha(s)\sin 2\pi\nu_0 \end{pmatrix}$$

- We calculate the trace to find the new tune

$$\cos 2\pi\nu = \frac{1}{2}Tr\left(\mathbf{M}'\right) = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$$

- For small errors

$$\cos 2\pi\left(\nu_0 + \Delta\nu\right) \approx \cos 2\pi\nu_0 - 2\pi\sin 2\pi\nu_0\Delta\nu = \cos 2\pi\nu_0 - \frac{1}{2f}\beta(s)\sin 2\pi\nu_0$$

$$\Rightarrow \Delta\nu = \frac{1}{4\pi}\frac{\beta(s)}{f}$$

Total Tune Shift

- The focal length associated with a local anomalous gradient is

$$d\left(\frac{1}{f}\right) = \frac{B'}{(B\rho)}ds$$

- So the total tune shift is

$$\Delta\nu = \frac{1}{4\pi}\oint \beta(s)\frac{B'(s)}{(B\rho)}ds$$

Local Correction

- Recall our generic transfer matrix

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_0\beta_1}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta_1}}\left((\alpha_0-\alpha_1)\cos\Delta\psi - (1+\alpha_0\alpha_1)\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta_1}}\left(\cos\Delta\psi - \alpha_1\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- If we use a dipole to introduce a small bend Θ at one point, it will in general propagate as

$$\begin{pmatrix} x(\Delta\psi) \\ x'(\Delta\psi) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_0\beta(s)}\sin\Delta\psi \\ \frac{1}{\sqrt{\beta_0\beta(s)}}\left((\alpha_0-\alpha(s))\cos\Delta\psi - (1+\alpha_0\alpha(s))\sin\Delta\psi\right) & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\psi - \alpha(s)\sin\Delta\psi\right) \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

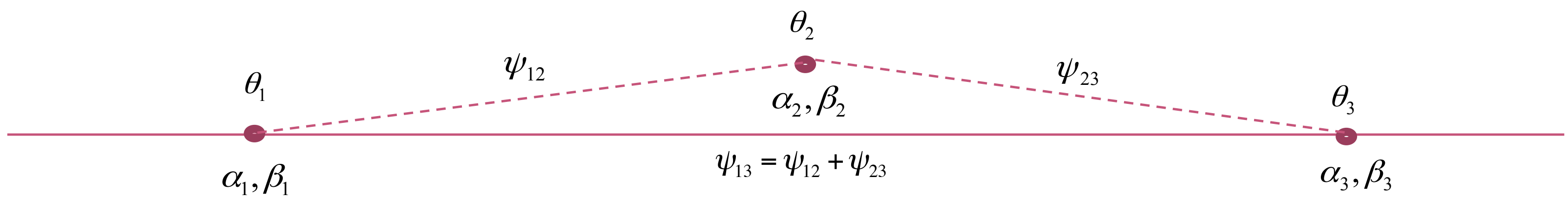
$$\boxed{x(\Delta\psi) = \theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi}$$

Remember this one forever

$$x'(\Delta\psi) = \theta\sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\psi - \alpha(s)\sin\Delta\psi\right)$$

"Three Bump"

- Consider a particle going down a beam line. By using a combination of three magnets, we can localize the beam motion to one area of the line



- We require

$$x_3 = \theta_1 \sqrt{\beta_1 \beta_3} \sin\psi_{13} + \theta_2 \sqrt{\beta_2 \beta_3} \sin\psi_{23} = 0$$

$$\Rightarrow \theta_2 = -\theta_1 \sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\theta_3 = -\left(\theta_1 \sqrt{\frac{\beta_1}{\beta_3}} \left(\cos\psi_{13} - \alpha_3 \sin\psi_{13} \right) + \theta_2 \sqrt{\frac{\beta_2}{\beta_3}} \left(\cos\psi_{23} - \alpha_3 \sin\psi_{23} \right) \right)$$

Cancel out angle from first two bends

$$= -\theta_1 \left(\sqrt{\frac{\beta_1}{\beta_3}} \left(\cos\psi_{13} - \alpha_3 \sin\psi_{13} \right) - \sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin\psi_{13}}{\sin\psi_{23}} \sqrt{\frac{\beta_2}{\beta_3}} \left(\cos\psi_{23} - \alpha_3 \sin\psi_{23} \right) \right)$$

$$= -\theta_1 \sqrt{\frac{\beta_1}{\beta_3}} \left(\cos\psi_{13} - \frac{\sin\psi_{13}}{\sin\psi_{23}} \cos\psi_{23} \right) = -\theta \sqrt{\frac{\beta_1}{\beta_3}} \left(\frac{\sin\psi_{23} \cos\psi_{13} - \cos\psi_{23} \sin\psi_{13}}{\sin\psi_{23}} \right) = -\theta_1 \sqrt{\frac{\beta_1}{\beta_3}} \left(\frac{\sin\left(\psi_{23} - \psi_{13}\right)}{\sin\psi_{23}} \right)$$

$$\Rightarrow \theta_3 = \theta_1 \sqrt{\frac{\beta_1}{\beta_3}} \left(\frac{\sin\psi_{12}}{\sin\psi_{23}} \right)$$

Local Bumps are an *extremely* powerful tool in beam tuning!!

Controls Example

- From Fermilab “Acnet” control system
  - The B:xxxx labels indicate individual trim magnet power supplies in the Fermilab Booster
  - Defining a “MULT: *N*” will group the *N* following magnet power supplies
  - Placing the mouse over them and turning a knob on the control panel will increment the individual currents according to the ratios shown in green

```
! INJECTION POSITION
 MULT     :6
-B:VL5T  [5]*2.45 473 f(t) values          4.933          Amps
-B:VL6T  [5]*1  6 473 f(t) values          2.117          Amps
-B:VL7T  [5]*2.47 473 f(t) values          2.058          Amps
-B:VL5T  *2.4 VL5 473 f(t) values          4.933          Amps
-B:VL6T  *1   VL6 473 f(t) values          2.117          Amps
-B:VL7T  *2.4 VL7 473 f(t) values          2.058          Amps
 MULT     :3
-B:VL5T  [1]*2.45 473 f(t) values          5.717          Amps
-B:VL6T  [1]*1  6 473 f(t) values          3.566          Amps
-B:VL7T  [1]*2.47 473 f(t) values          2.561          Amps
 MULT     :3
-B:VL5T  [2]*2.45 473 f(t) values          5.642          Amps
-B:VL6T  [2]*1  6 473 f(t) values           .427          Amps
-B:VL7T  [2]*2.47 473 f(t) values           .718          Amps
 MULT     :3
-B:VL5T  [3]*2.45 473 f(t) values          20.65          Amps
-B:VL6T  [3]*1  6 473 f(t) values          3.389          Amps
-B:VL7T  [3]*2.47 473 f(t) values          9.95           Amps
 MULT     :3
-B:VL5T  [4]*2.45 473 f(t) values          15.21          Amps
-B:VL6T  [4]*1  6 473 f(t) values          6.348          Amps
-B:VL7T  [4]*2.47 473 f(t) values          16.35          Amps
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Off-Momentum Particles

- Our previous discussion implicitly assumed that all particles were at the same momentum
  - Each quad has a constant focal length
  - There is a single nominal trajectory
- In practice, this is never true. Particles will have a distribution about the nominal momentum
- We will characterize the behavior of off-momentum particles in the following ways
  - "Dispersion" (*D*): the dependence of position on deviations from the nominal momentum

$$\Delta x(s) = D_x(s)\frac{\Delta p}{p_0}$$

D has units of length

  - "Chromaticity" (η) : the change in the tune caused by the different focal lengths for off-momentum particles

$$\Delta \nu_x = \xi_x \frac{\Delta p}{p_0} \quad \left(\text{sometimes } \frac{\Delta \nu_x}{\nu_x} = \xi_x \frac{\Delta p}{p_0}\right)$$

  - Path length changes (momentum compaction)

$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

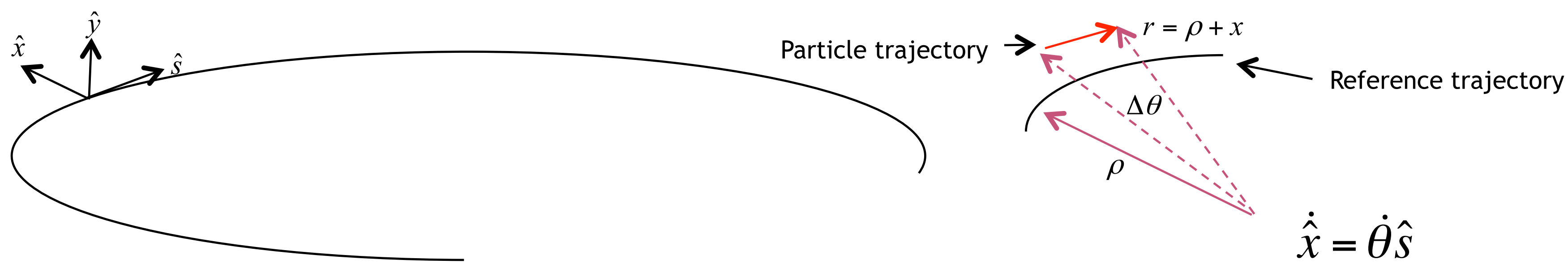
Equations of Motion (redoing the steps we skipped)

- General equation of motion

$$\vec{F} = e\vec{v}\times\vec{B} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\gamma m\dot{\vec{R}} = \gamma m\ddot{\vec{R}}$$

$$\Rightarrow \ddot{\vec{R}} = \frac{e\vec{v}\times\vec{B}}{\gamma m} = \frac{e}{\gamma m}\begin{vmatrix}\hat{x} & \hat{y} & \hat{s}\\ v_x & v_y & v_s\\ B_x & B_y & 0\end{vmatrix} = \frac{e}{\gamma m}\left(-v_sB_y\hat{x} + v_sB_x\hat{y} + \left(v_xB_y - v_yB_x\right)\hat{s}\right)$$

- For the moment, we will consider motion in the horizontal (*x*) plane, with a reference trajectory established by the dipole fields.



$$\dot{\hat{x}} = \dot{\theta}\hat{s}$$

$$\dot{\hat{s}} = -\dot{\theta}\hat{x}$$

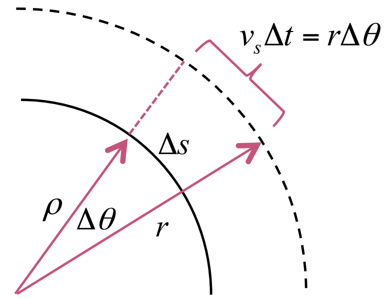
- Solving in this coordinate system, we have

$$\begin{aligned}
\vec{R} &= r\hat{x} + y\hat{y} \\
\dot{\vec{R}} &= \dot{r}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y} &&= \dot{r}\hat{x} + r\dot{\theta}\hat{s} + \dot{y}\hat{y} \\
\ddot{\vec{R}} &= \left(\ddot{r}\hat{x} + \dot{r}\dot{\hat{x}}\right) + \left(\dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}\dot{\hat{s}}\right) + \ddot{y}\hat{y} &&= \ddot{r}\hat{x} + 2\dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} - r\dot{\theta}^2\hat{x} + \ddot{y}\hat{y} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{x} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{s} + \ddot{y}\hat{y}
\end{aligned}$$

Equations of Motion (cont'd)

- Equating the *x* terms

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s B_y}{\gamma m} = -\frac{e v_s^2 B_y}{\gamma m v_s} = -\frac{e v_s^2 B_y}{p} = -\frac{v_s^2 B_y}{(B\rho)}$$



Note: *s* measured along *nominal* trajectory, v_s measured along *actual* trajectory

$$\dot{s} = \frac{ds}{dt} = \frac{\rho}{r} v_s$$

- Re-express in terms of path length *s*. Use

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_s \frac{\rho}{r}\frac{d}{ds} \Rightarrow \frac{d^2}{dt^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}; \qquad \dot{\theta} = \frac{v_s}{r}$$

- Rewrite equation

$$\left(v_s \frac{\rho}{r}\right)^2 \frac{d^2 r}{ds^2} - r\left(\frac{v_s}{r}\right)^2 = -\frac{v_s^2 B_y}{(B\rho)}$$

$$\underset{\text{(rearrange terms)}}{\Rightarrow} \qquad r'' = -\frac{v_s^2 B_y}{(B\rho)}\frac{r^2}{\rho^2} + \frac{r}{\rho^2}$$

$$\underset{(\text{use } r = \rho + x)}{\Rightarrow} \qquad x'' = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2}; \qquad y'' = \frac{B_x}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2$$

Treating off-momentum Particle Motion

- To treat off-momentum particles, we start with our original these original equations of motion.

$$x'' = -\frac{B_y}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2}; \qquad y'' = \frac{B_x}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2$$

- The rigidity shown is for the nominal momentum, but I can correct the equation for the true momentum by substituting the "true" rigidity

$$(B\rho)_{true} = (B\rho)\frac{p}{p_0} \rightarrow \frac{1}{(B\rho)_{true}} = \frac{1}{(B\rho)}\frac{p_0}{p} = \frac{1}{(B\rho)}\frac{p_0}{(p_0+\Delta p)} \approx \frac{1}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)$$

- If we substitute that into the equation, and keep only the lowest order terms, we have

$$x'' = -\frac{B_y}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2} = (\ldots) + \frac{B_y}{(B\rho)}\frac{\Delta p}{p_0} \equiv (\ldots) + \frac{B_y}{(B\rho)}\delta$$

$$y'' = \frac{B_x}{(B\rho)}\left(1-\frac{\Delta p}{p_0}\right)\left(1+\frac{x}{\rho}\right)^2 = (\ldots) - \frac{B_x}{(B\rho)}\delta$$

- If we then also keep only the lowest order terms in the fields, we have

$$B_y = B_0 + B'x \approx B_0; \quad B_x = B'y \approx 0$$

- If we take only the first order terms in the rest of the equation, as we did before, the only change is that extra term on the RHS.

$$x'' + \left(\frac{1}{\rho^2} + \frac{1}{(B\rho)}B'\right)x = \frac{B_0}{(B\rho)}\delta = \frac{1}{\rho}\delta; \qquad y'' - \frac{1}{(B\rho)}B'y = 0$$

- This is a second order differential inhomogeneous differential equation, so the solution is

$$x(s) = x_0 C(s) + x_0' S(s) + \delta d(s)$$

$$x'(s) = x_0 C'(s) + x_0' S'(s) + \delta d'(s)$$

Where d(s) is the solution particular solution of the differential equation

$$d'' + Kd = \frac{1}{\rho}$$

- We solve this piecewise, for *K* constant and find

$$K > 0: \quad d(s) = \frac{1}{\rho K}\left(1 - \cos\sqrt{K}s\right)$$

$$d'(s) = \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s$$

$$K < 0: \quad d(s) = -\frac{1}{\rho K}\left(1 - \cosh\sqrt{K}s\right)$$

$$d'(s) = \frac{1}{\rho\sqrt{K}}\sinh\sqrt{K}s$$

- Recall

$$x'' + \left(\frac{1}{\rho^2} + B_y'\right)x \Rightarrow K = \left(\frac{1}{\rho^2} + B_y'\right)$$

Matrix Representation

- I have the solution

$$x(s) = x_0 C(s) + x_0' S(s) + \delta d(s)$$
$$x'(s) = x_0 C'(s) + x_0' S'(s) + \delta d'(s)$$

Solution to the on-momentum case

Off-momentum correction

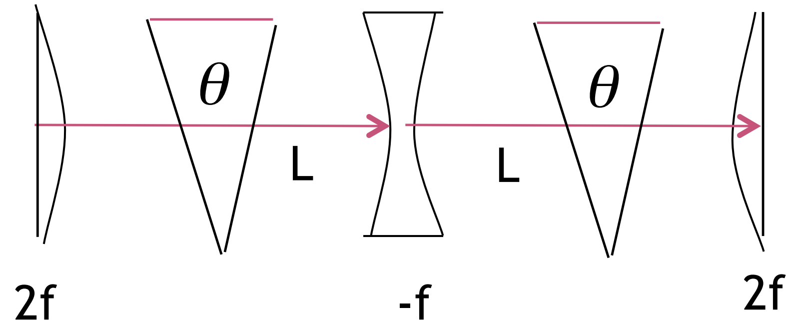
- We can express this in matrix form as

Usual transfer matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & d(s) \\ m_{21} & m_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

Example: FODO Cell

- We look at our symmetric FODO cell, but assume that the drifts are bend magnets



For a thin lens *d~d'~0*. For a pure bend magnet

$s << \rho$

$$K = \frac{1}{\rho^2}: \quad d(s) = \frac{1}{\rho K}\left(1-\cos\sqrt{K}s\right) = \rho\left(1-\cos\frac{s}{\rho}\right) \approx \frac{1}{2\rho}s^2 \rightarrow \frac{1}{2}\frac{L^2}{\rho} = \frac{1}{2}\theta L$$

$$d'(s) = \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s = \sin\frac{s}{\rho} \approx \frac{s}{\rho} \rightarrow \theta$$

Leading to the transfer Matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & L & \frac{L\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & L & \frac{L\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L\left(1+\frac{L}{2f}\right) & 2L\theta\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta\left(1-\frac{L}{4f}-\frac{L^2}{8f^2}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

Solving for Equilibrium Orbit

- We hypothesize that we will have a new equilibrium reference orbit for off-momentum particles, defined as a function of position by

$$x(s,\delta) = D_x(s)\delta$$

"Dispersion"

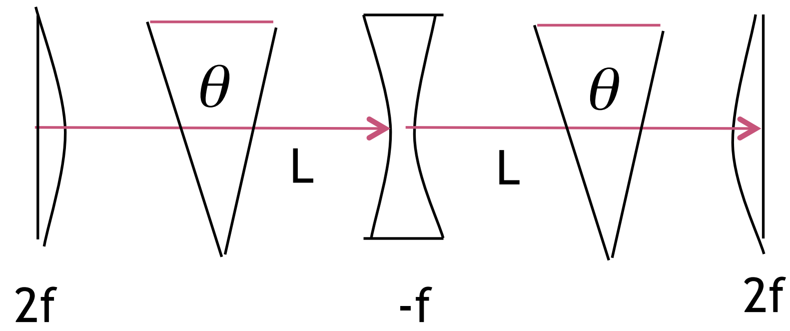
- We expect this to have the periodicity of the lattice, so in general

$$\begin{pmatrix} \delta D_x \\ \delta D'_x \\ \delta \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu & d \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu & d' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta D_x \\ \delta D'_x \\ \delta \end{pmatrix}$$

Can set δ to 1

Solving for Dispersion

- Going back to our original FODO cell (with bends)



- We must solve

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L\left(1+\frac{L}{2f}\right) & 2L\theta\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta\left(1-\frac{L}{4f}-\frac{L^2}{8f^2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

Solving for Lattice Functions

- We have already solved for the basic lattice functions

$$\sin\frac{\mu}{2} = \frac{L}{2f}$$

$$\beta_{F,D} = \frac{2L\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu}$$

- You'll solve in the homework

$$D_{F,D} = \frac{\theta L\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

- It will really simplify things if you invoke symmetry to show that

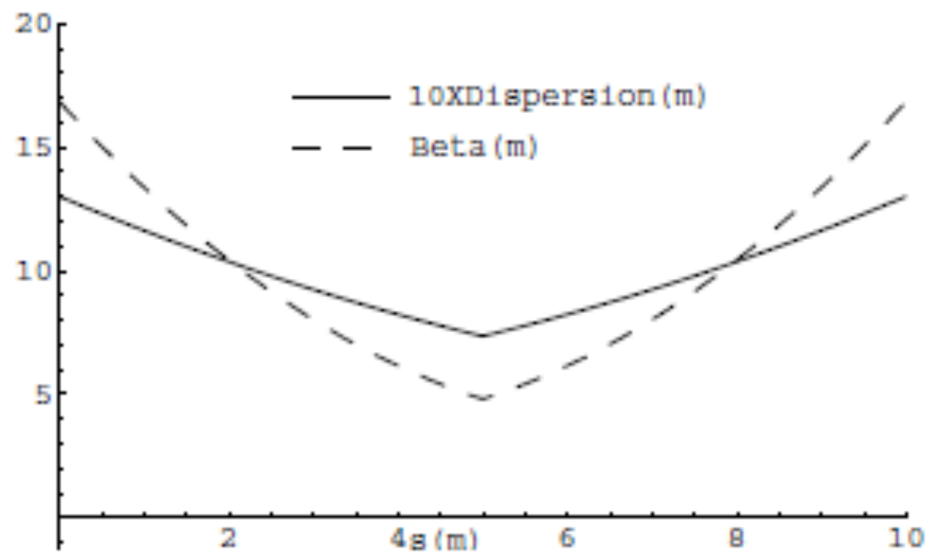
$$\alpha_{F,D} = D'_{F,D} = 0$$

Evolution of Dispersion Functions

- Since the dispersion functions represent displacements, they will evolve like the position

$$\begin{pmatrix} D_x(s) \\ D'_x(s) \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & d(s) \\ m_{21} & m_{22} & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_x(0) \\ D'_x(0) \\ 1 \end{pmatrix}$$

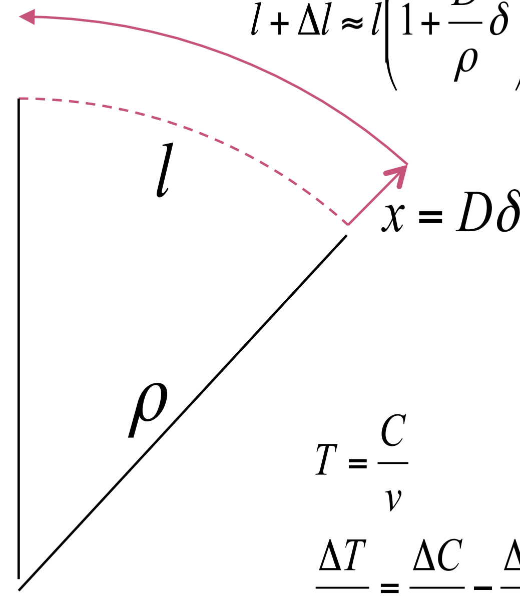
- Putting it all together



Momentum Compaction and Slip Factor

- In general, particles with a high momentum will travel a longer path length. We have

$$l + \Delta l \approx l\left(1 + \frac{D}{\rho}\delta\right)$$



$$C(p_0) = \oint ds$$

$$C(p_0 + \Delta p) = \oint \left(1 + \frac{D}{\rho}\frac{\Delta p}{p_0}\right) ds$$

$$\frac{\Delta C}{C} = \frac{\oint \frac{D}{\rho} ds}{\oint ds}\delta = \left\langle \frac{D}{\rho} \right\rangle \delta$$

"momentum compaction" → $\equiv \alpha\delta$

$$T = \frac{C}{v}$$

$$\frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = \frac{\Delta C}{C} - \frac{\Delta \beta}{\beta}$$

$$= \alpha \frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

$$= \left(\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

$$\equiv \eta \frac{\Delta p}{p}$$

- The slip factor is defined as the fractional change in the orbital period
- Note that

$\gamma < \gamma_T: \quad \eta < 0 \quad$ higher energy particles take *less* time to go around

$\gamma > \gamma_T: \quad \eta > 0 \quad$ higher energy particles take *more* time to go around

$\gamma = \gamma_T: \quad \eta = 0 \quad$ "transition"

Transition γ

- In homework, you showed that for a simple FODO CELL

$$\beta_{\max,\min} = 2L\frac{\left(1 \pm \sin\frac{\mu}{2}\right)}{\sin\mu}; \text{ and } D_{\max,\min} = \theta L\frac{\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

- If we assume they vary ~linearly between maxima, then for small μ

$$\langle\beta\rangle \approx \frac{2L}{\mu}; \ \langle D\rangle \approx \frac{4\theta L}{\mu^2} = 4\frac{L^2}{\mu^2\rho} = \frac{\langle\beta\rangle^2}{\rho}$$

- It follows

$$\nu \approx \frac{R}{\langle\beta\rangle} \approx \frac{\rho}{\langle\beta\rangle} \Rightarrow \langle D\rangle \approx \frac{\rho}{\nu^2}$$

$$\alpha_C = \frac{1}{C}\oint\frac{D}{\rho}ds \approx \frac{1}{\rho}\langle D\rangle \approx \frac{1}{\nu^2}$$

$$\gamma_t = \frac{1}{\sqrt{\alpha_C}} \approx \nu$$

- This approximation generally works better than it should

Chromaticity

- In general, momentum changes will lead to a tune shift by changing the effective focal lengths of the magnets

$$\frac{1}{f}=\frac{B'l}{(B\rho)}=\frac{B'l}{(B\rho)_0}\frac{p_0}{p}\approx\frac{1}{f_0}\left(1-\frac{\Delta p}{p_0}\right)$$

$$\Delta\frac{1}{f}=-\frac{1}{f_0}\frac{\Delta p}{p_0}$$

$$\Rightarrow\Delta\nu=-\frac{1}{4\pi}\sum_i\beta_i\frac{1}{f_i}\frac{\Delta p}{p_0}\equiv\xi\frac{\Delta p}{p_0}$$

- As we are passing trough a magnet, we can find the focal length as

$$\frac{1}{f_0}=\int_0^L\frac{B'}{(B\rho)}ds$$

- But remember that our general equation of motion is

$$x''+\left(\frac{1}{\rho^2}+\frac{B'(s)}{(B\rho)}\right)x=0\equiv x''+K(s)x$$

- Clearly, a change in momentum will have the same effect on the entire focusing term, so we can write in general

$$\xi=-\frac{1}{4\pi}\oint\beta_0(s)K(s)ds$$

- This can be expressed in terms of lattice functions (after a lot of messy algebra) as

$$\xi=-\frac{1}{4\pi}\oint\left(\gamma(s)+\alpha'(s)\right)ds$$

Chromaticity and Sextupoles

- we can write the field of a sextupole magnet as

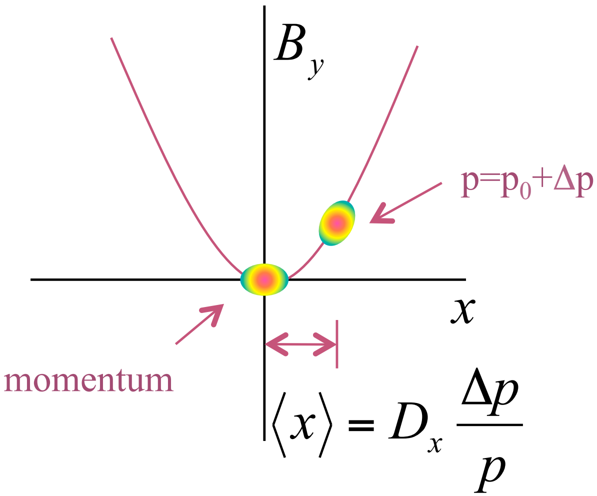
$$B(x) = \frac{1}{2}B''x^2 \qquad \left(\text{often expressed } b_2 x^2\right)$$

- If we put a sextupole in a dispersive region then off momentum particles will see a gradient

$$B'\left(x = D\delta\right) \approx B''D\frac{\Delta p}{p_0}$$

which is effectively like a position dependent quadrupole, with a focal length given by

$$\frac{1}{f_{eff}} = \frac{B''}{(B\rho)}LD\frac{\Delta p}{p_0}$$



- So we write down the tune-shift as

$$\Delta\nu = \frac{1}{4\pi}\beta\frac{1}{f_{eff}} = \frac{1}{4\pi}\frac{\beta B''}{(B\rho)}LD\frac{\Delta p}{p_0} \equiv \xi\frac{\Delta p}{p_0}$$

$$\Rightarrow \xi_S = \frac{1}{4\pi}\frac{\beta B''}{(B\rho)}LD$$

- Note, this is only valid when the motion due to momentum is large compared to the particle spread (homework problem)