E&M and Relativity

This Lecture

- Math Refresher (Expectations)
- Maxwell's Equations
- Special Relativity
- Multipole Expansion of Magnetic Fields

Expectations: Basics and Refreshers

- Matrix Operations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} aV_1 + bV_2 \\ cV_1 + dV_2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

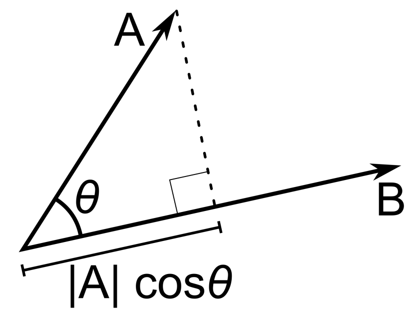
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Expectations (cont'd)

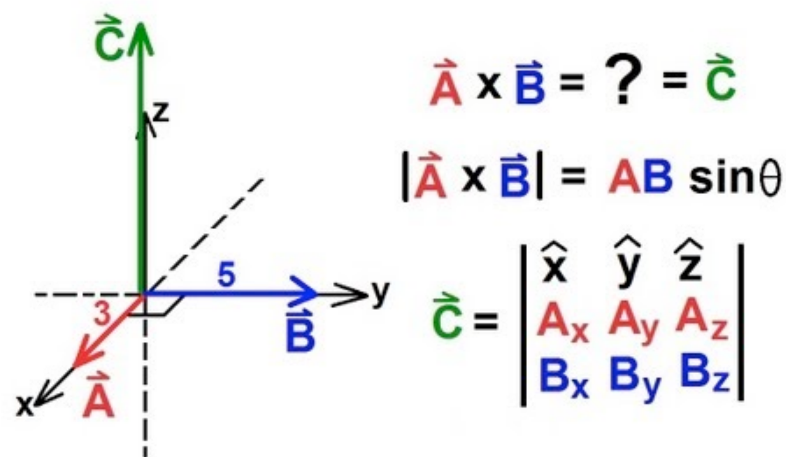
- Vector Operations
  - Dot product

$$\vec{A}\cdot\vec{B} = (A_xB_x + A_yB_y + A_zB_z)$$



  - Cross product

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Expectations (cont'd)

- Vector differential operations
  - Grad operator

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

  - Gradient

$$\vec{\nabla}\phi \equiv \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \right)$$

  - Divergence

$$\vec{\nabla} \cdot \vec{A} \equiv \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

  - Curl

$$\vec{\nabla} \times \vec{A} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)\hat{j} + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)\hat{k}$$

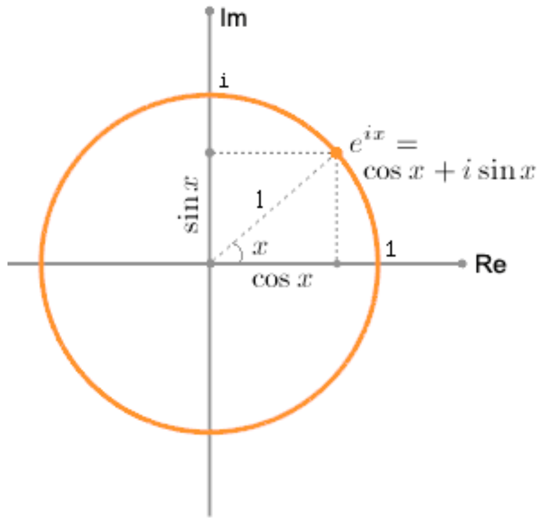
Euler Relations

- You should be very comfortable with the complex plane

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$



- Also remember the Taylor expansions of trig functions

$$e^{\theta} \approx 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$$

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

Some Handy Relationships

- Memorize these because we'll use them a lot!

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$
$$\cos A\sin B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$$
$$\cos A\cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$
$$\sin A\sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\cos^2 A = \frac{1}{2}\left(1 + \cos(2A)\right)$$
$$\sin^2 A = \frac{1}{2}\left(1 - \cos(2A)\right)$$

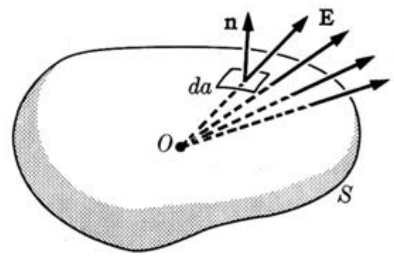
Maxwell's Equations

- In 1861, James Maxwell began his attempt to find a self-consistent set of equations consistent with all of the E&M experiments which had been done up until that point.
  - Because vector calculus hadn't been invented yet, his final paper is 55 pages long and completely incomprehensible.
- In in modern notation, it reduces to the following four equations:

$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_0}$	$\Rightarrow \oint_S \vec{E}\bullet d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$	Gauss' Law
$\vec{\nabla}\bullet\vec{B} = 0$	$\Rightarrow \oint_S \vec{B}\bullet d\vec{A} = 0$	No Name Law
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\Rightarrow \oint_C \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B}\bullet d\vec{A}$	Faraday's Law
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$	$\Rightarrow \oint_C \vec{B}\bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\oint_S \vec{E}\bullet d\vec{A}$	Ampere's Law

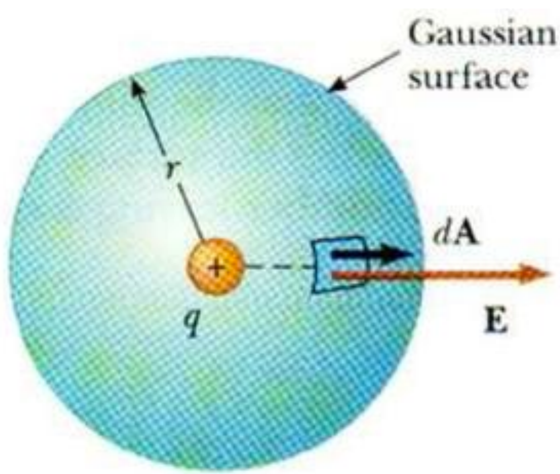
Gauss' Law

- The electric field passing through a surface depends only on the charge contained within the surface



$$\oint_S \vec{E} \bullet d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

- Example: deriving Coulomb's Law



$$\oint_S \vec{E} \bullet d\vec{A} = E \bullet A$$

$$= 4\pi r^2 E \quad \rightarrow E = \frac{q}{4\pi r^2 \epsilon_0}$$

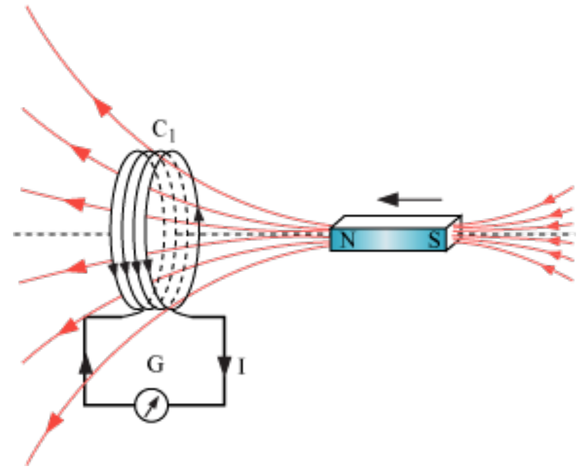
$$= \frac{q}{\epsilon_0}$$

$$\oint_S \vec{B} \bullet d\vec{A} = 0 \rightarrow \text{No magnetic monopoles}$$

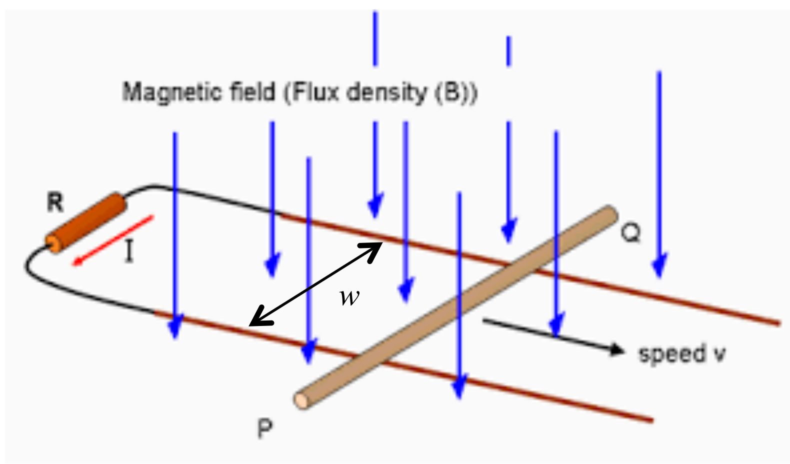
Faraday's Law

- The integrated electric field around any closed loop is proportional to the rate of change of the magnetic flux passing through the loop

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B} \bullet d\vec{A}$$



- Example: magnetic induction



$$\begin{aligned} V &= \oint_C \vec{E} \bullet d\vec{l} \\ &= -\frac{\partial}{\partial t}\oint_S \vec{B} \bullet d\vec{A} \\ &= -B\frac{dA}{dt} \\ &= -Bwv \end{aligned}$$

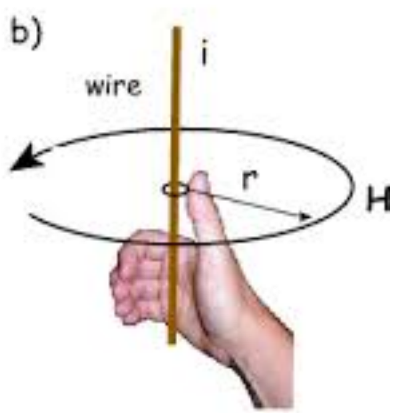
Ampere's Law

- The integrated magnetic field around any closed loop is proportional to the total current passing through the loop.

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \oint_S \vec{E} \bullet d\vec{A}$$

Set to 0 for a minute

- Example: Magnetic field of a wire



$$\oint_C \vec{B} \bullet d\vec{l} = 2\pi r B$$

$$= \mu_0 I_{enclosed} = \mu_0 I$$

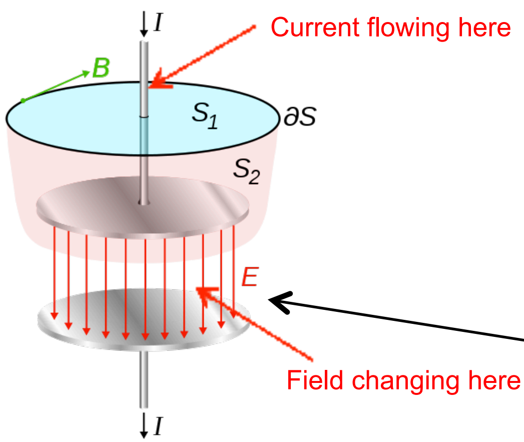
$$\rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Displacement Current

- Maxwell's first version of Ampere's Law did not have the second term

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed}$$

- However, you should be able to draw the surface anywhere, and you get in trouble if you draw it through a break in the current



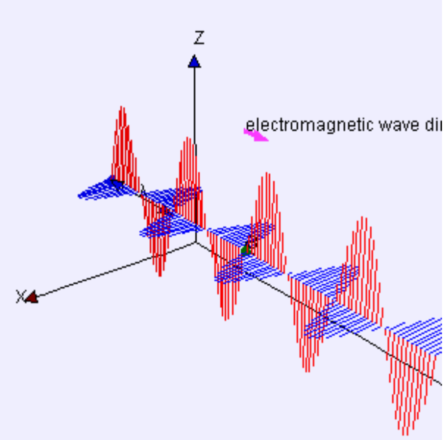
However, anywhere there's a break in the current, you'll get a *changing electric field*.

- Maxwell added the second term *just so he would get the same answer in both cases!*

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \oint_S \vec{E} \bullet d\vec{A}$$

Electromagnetic Waves

- The “displacement current” was added for purely mathematical reasons
  - It would not be proven experimentally for many years
- However, the implications were profound
- Previously, it was believed you could not have electric or magnetic fields without electric charges, but now, even in a complete vacuum, you can have
  - (changing electric field)➔(changing magnetic field)➔ (changing electric field)➔“Electromagnetic Wave”!
- Moreover, Maxwell could calculate the velocity, and he found it was the speed of light!

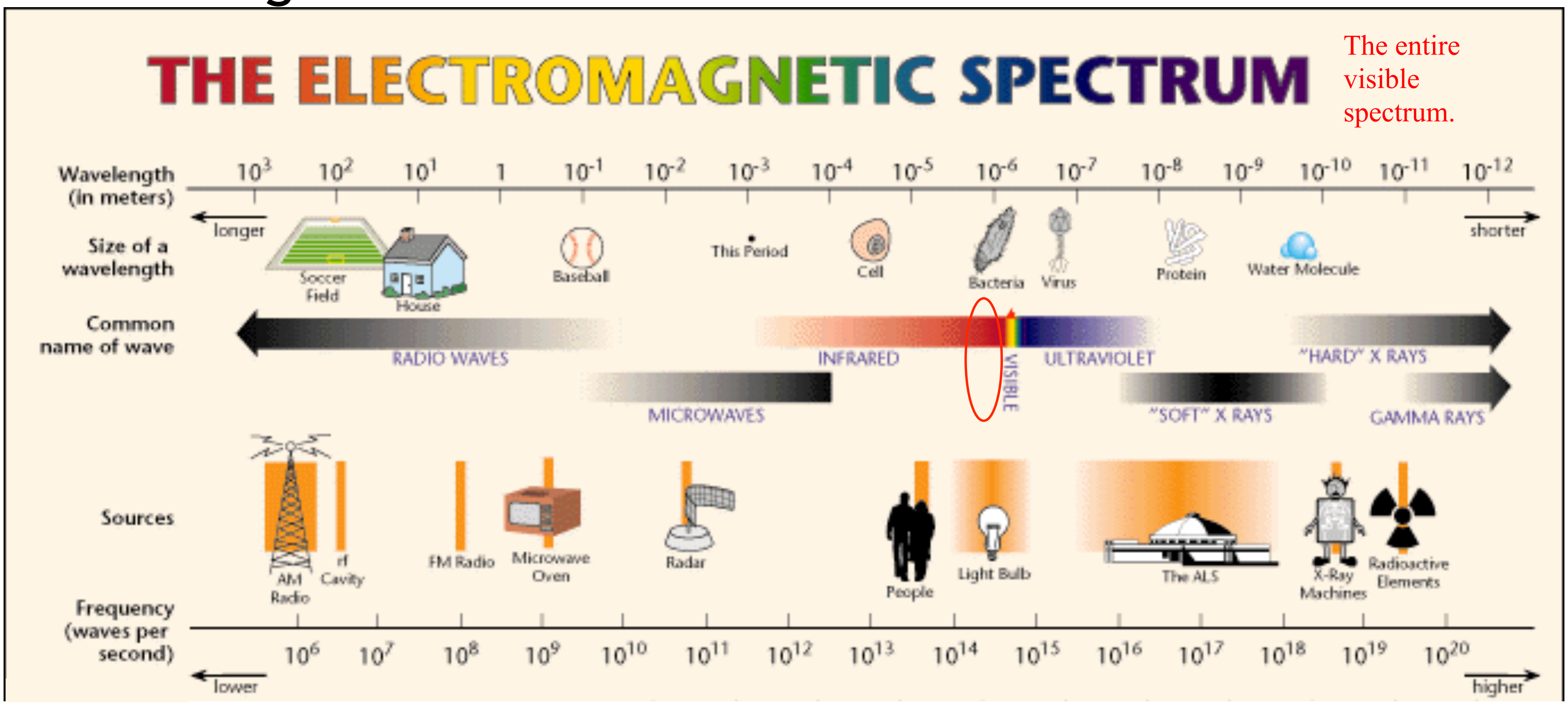


- He wrote (with trembling hands, maybe?)

"we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena"

It's all the same thing…

- In one fell swoop, Maxwell not only unified electricity and magnetism, but his results would eventually show that light, heat, radio waves, x-rays, gamma rays, etc., are *all* really the same thing – differing only in wavelength!

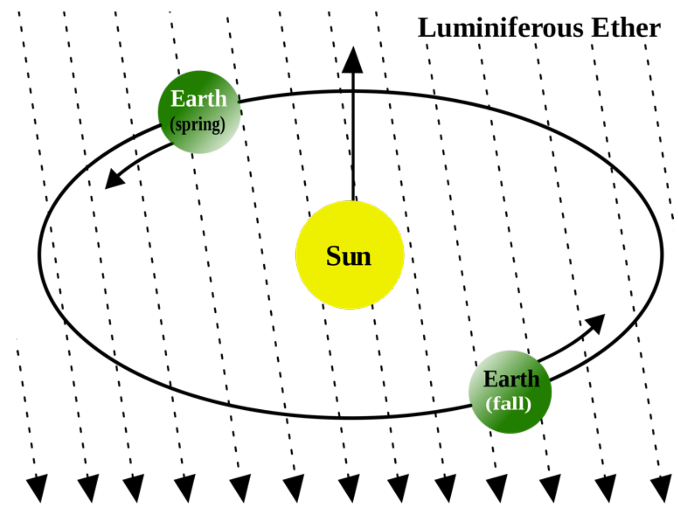


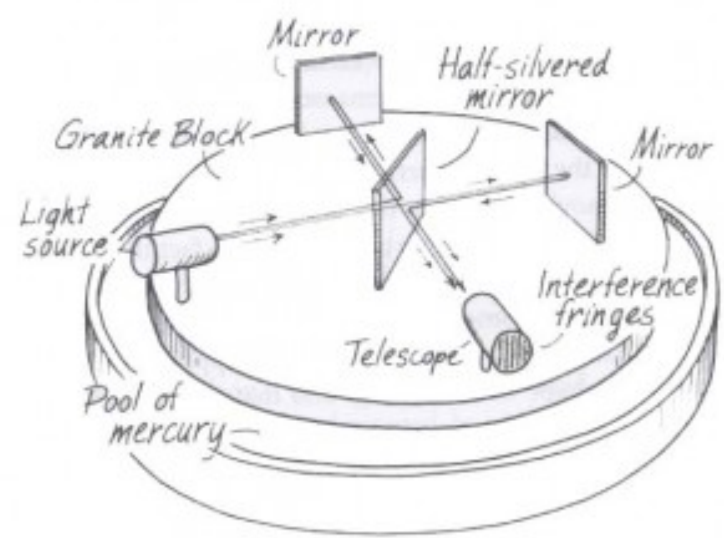
What's "undulating"?

- As often happens science, one answer raised a lot more questions.
- All (other) known waves require a "medium" (air, water, earth, "the wave") to travel through.
- Light at least appears to travel through a vacuum.
- In science, always try the simplest answer first:
  - Maybe vacuum isn't really empty?
- Scientists hypothesized the existence of "luminiferous aether", and started to look for it…

Michelson-Morley Experiment

- If aether exists, then it must fill space and the earth must be passing through it.
- Light traveling along the direction of the Earth's motion should have a *slightly different* wavelength than light traveling transverse to it.
- In 1887, Albert Michelson and Edward Morley performed a sensitive experiment to measure this difference.
- Their result:
  - No difference → no aether!
- Biggest mystery in science for almost 20 years.



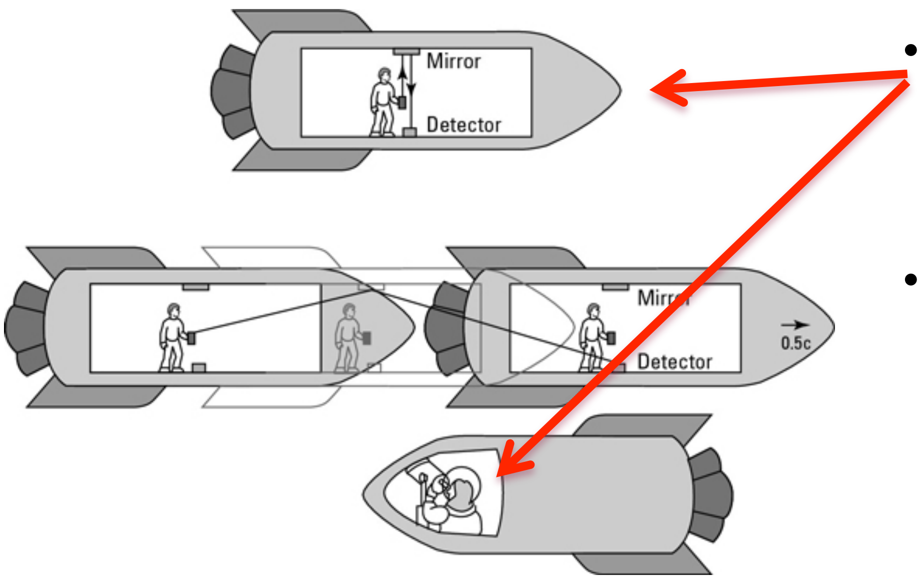


Einstein to the Rescue

- In 1905, Albert Einstein postulated that perhaps the equations meant exactly what they appeared to mean:
  - The speed of light was the same *in any frame* in which is was measured.
- He showed that this could “work”, but only if you gave up the notion of fixed time.
  - ➔ “Special Theory of Relativity”
- Profound implications…

Example: Time Dilation

- Einstein said, "The speed of light must be the same in any reference frame". For example, the time it takes light to bounce off a mirror in a spaceship must be the same whether it's measured by someone in the spaceship, or someone outside of the spaceship.

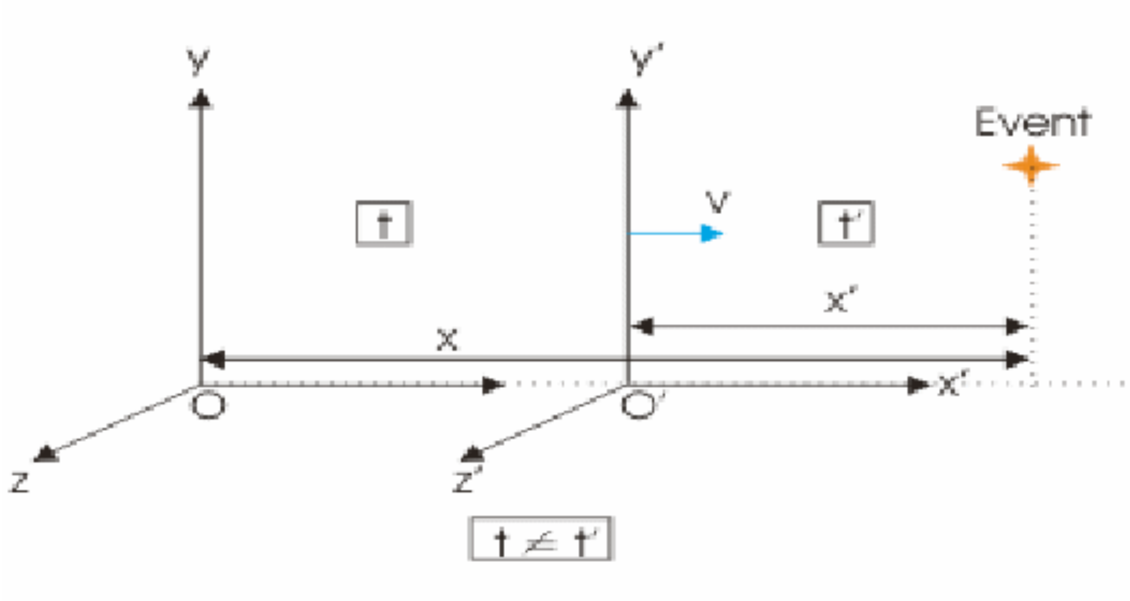


- These two people have to measure *the same* speed for light, even though light is traveling a different distance for the two of them.
- The only solution? More time passes for the stationary observer than the guy in the spaceship!
    - "Twin Paradox"

- This seems weird, but it applies to everything we do at the lab
    - Example: the faster pions and muons move, the longer they live.

Lorentz Transformations

- Generally, relativity treats time more or less like one more spatial dimension. Both time and space transform between two frames



$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

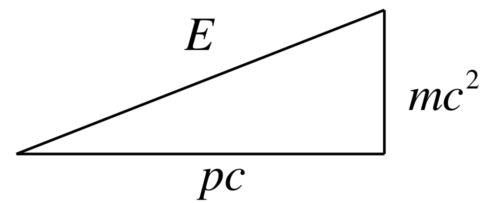
Momentum and Energy in Special Relativity

- Classically:

$$\text{momentum: } \vec{p} = m\vec{v}$$

$$\text{kinetic energy: } K = \frac{1}{2}mv^2$$

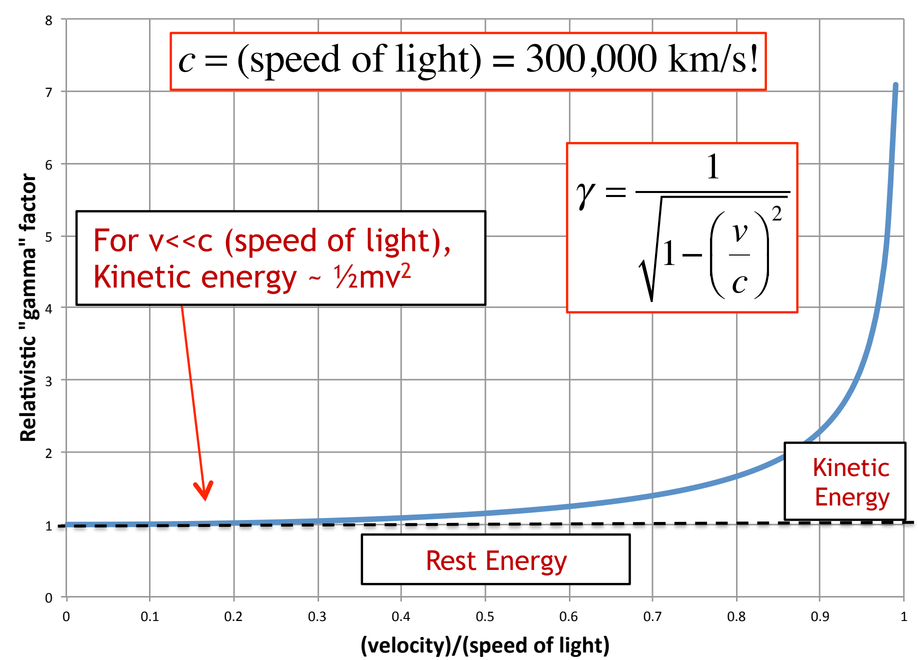
- Relativistically:



$$\text{momentum: } \vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}$$

$$\text{total energy: } E^2 = (mc^2)^2 + (pc)^2$$

$$\text{kinetic energy: } K = E\text{-}mc^2$$



Notation and Formalism

- Basics

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\text{momentum } p = \gamma m v$$

$$\text{total energy } E = \gamma m c^2$$

$$\text{kinetic energy } K = E - mc^2$$

$$E = \sqrt{\left(mc^2\right)^2 + \left(pc\right)^2}$$

Some Handy Relationships (homework)

$$\beta = \frac{pc}{E}$$

$$d\gamma = \beta\gamma^3 d\beta$$

$$\frac{d\beta}{\beta} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E}$$

- A word about units
  - For the most part, we will use SI units, except
    - Energy: eV (keV, MeV, etc) [1 eV = $1.6\text{x}10^{-19}$ J]
    - Mass: eV/c^2 [proton = $1.67\text{x}10^{-27}$ kg = 938 MeV/c^2]
    - Momentum: eV/c [proton @ β=.9 = 1.94 GeV/c]

4-Vectors and Lorentz Transformations

- We'll use the conventions

$$\mathbf{X} \equiv (x, y, z, ct)$$

$$\mathbf{P} \equiv \left(p_x, p_y, p_z, \frac{E}{c} \right)$$

$$\mathbf{A}' = \mathbf{\Lambda A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta \\ 0 & 0 & -\beta & \gamma \end{pmatrix} \mathbf{A} \quad \text{(velocity along z axis)}$$

$$|\mathbf{X}|^2 = (ct)^2 - x^2 - y^2 - z^2 \equiv (c\tau)^2$$

$$|\mathbf{P}|^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 \equiv (mc^2)^2$$

- Note that for a system of particles $\left|\sum \mathbf{P}_i\right|^2 = (M_{eff} c^2)^2 \equiv s$

- We'll worry about field transformations later, as needed

Back to Maxwell's Equation: EM Fields in Matter

- The equations we've talked about so far are correct if you account for all electric charges in the system; however, in real life situation, much, or even most, of the charge is a system is contained in matter, and it's behavior can generally be parameterized in a more convenient way. In terms of just the *free* electric charge, Gauss' Law and Ampere's Law become:

$$\vec{\nabla}\bullet\vec{D} = \rho_f \Rightarrow \oint_S \vec{D}\bullet d\vec{A} = Q_{f,enc} \qquad ;\vec{D}\equiv\varepsilon\vec{E}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \Rightarrow \oint_C \vec{H}\bullet d\vec{l} = I_{f,enclosed} +_0 \frac{\partial}{\partial t}\oint_S \vec{D}\bullet d\vec{A} \qquad ;\vec{H}\equiv\frac{\vec{B}}{\mu}$$

Local effects of media

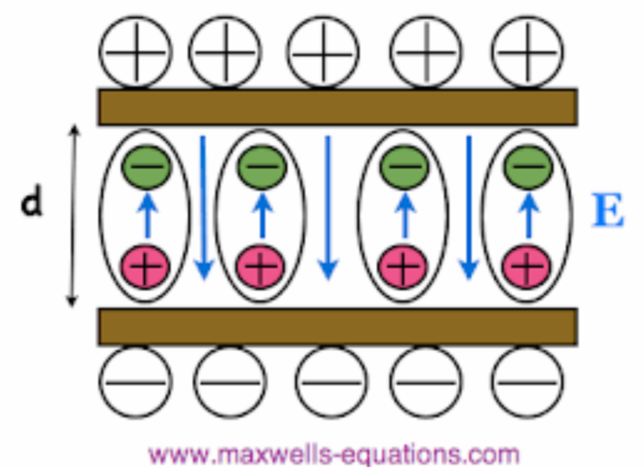
where

$$\epsilon = \text{"electric permitivity"}$$

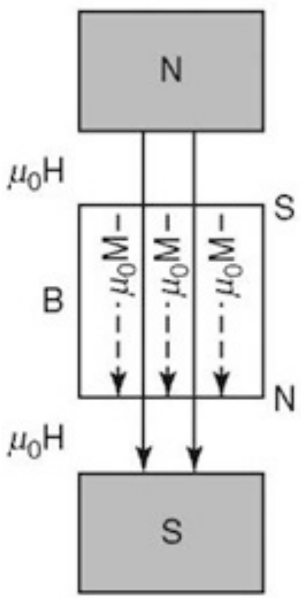
$$\mu = \text{"magnetic permiability"}$$

Fields in Matter

- The "electric permittivity" comes from the tendency of charge in matter to form electric dipoles in the presence of an external field, *reducing* the the true field

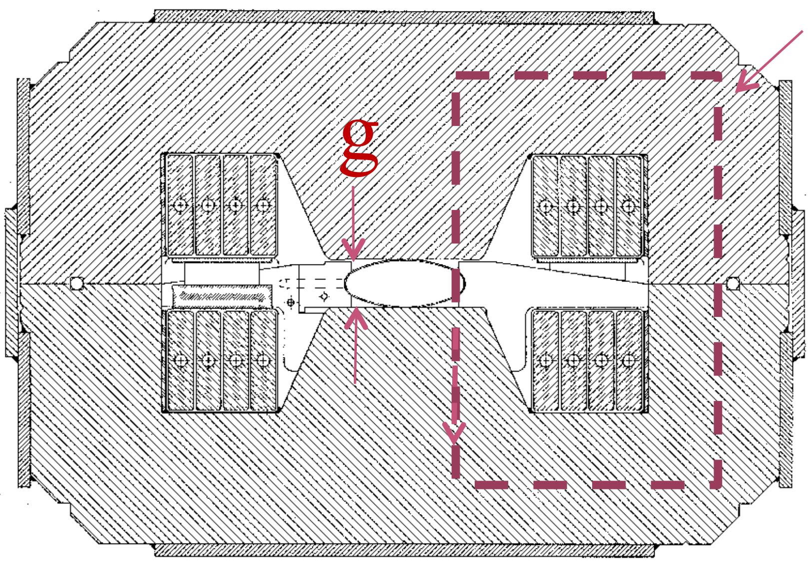


- The "magnetic permeability" comes from the tendency of magnetic dipoles in some materials to align with the external magnetic field, *increasing* the true field.



Example: Field in a permeable dipole

- Cross section of dipole magnet



$$\oint_C \vec{H} \bullet d\vec{l} = \frac{1}{\mu_{steel}} \int_{\text{path in steel}} \vec{B} \bullet d\vec{l} + \frac{B_{gap} g}{\mu_0}$$

$$\approx \frac{B_{gap} g}{\mu_0} = I_{enclosed}$$

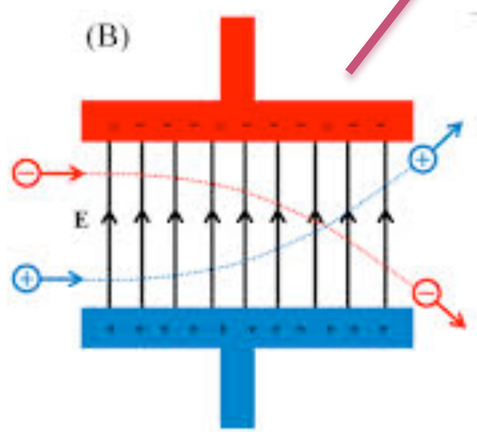
$\mu_{steel} \gg \mu_{gap}$

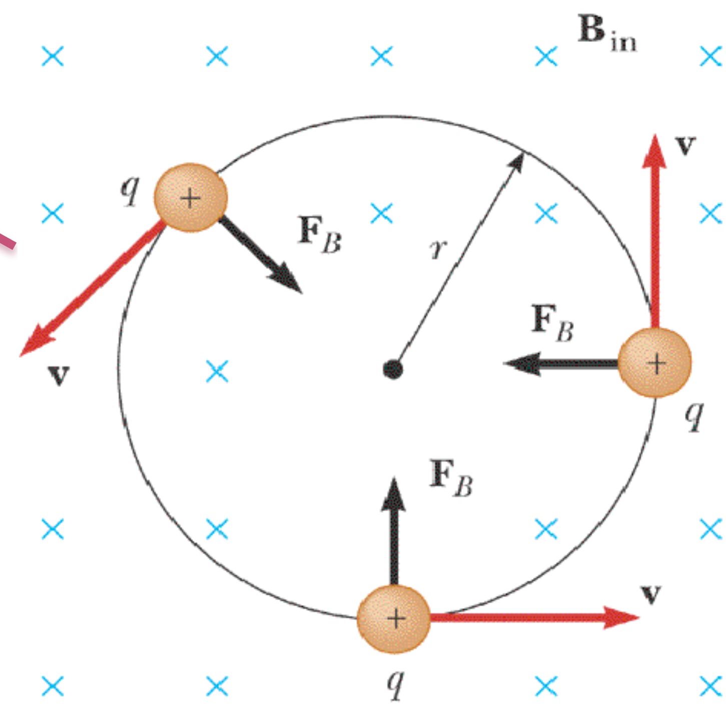
$$\Rightarrow B_{gap} \approx \frac{\mu_0 N_{turns} I}{g}$$

Particle Motion in EM Fields

- The relativistially correct form for the motion of charged particles in electric and magnetic fields is given by the Lorentz equation:

$$F = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$





radius of curvature $r = \frac{p}{qB}$

Cyclotron (1930's)

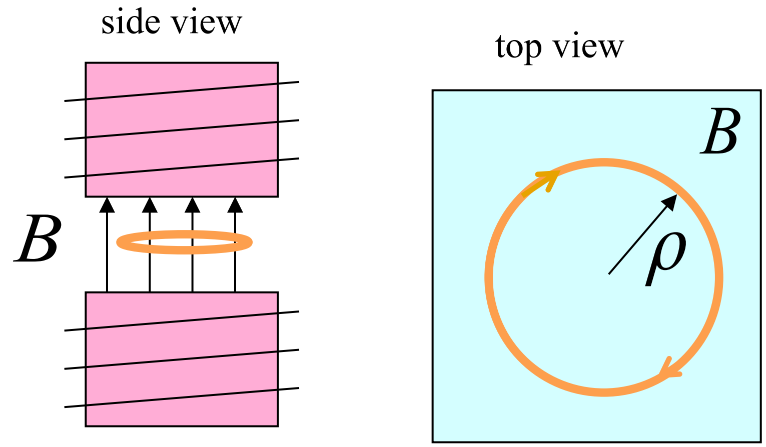
- A charged particle in a uniform magnetic field will follow a circular path of radius

$$\rho = \frac{mv}{qB} \quad (v << c)$$

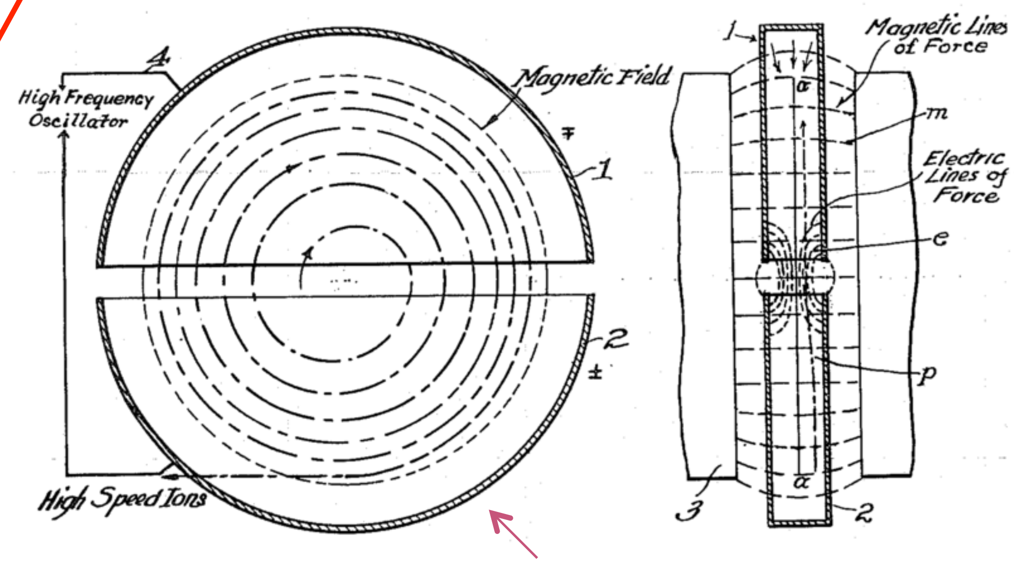
$$f = \frac{v}{2\pi\rho}$$

$$= \frac{qB}{2\pi m} \quad \text{(constant!!)}$$

$$\Omega_s = 2\pi f = \frac{qB}{m}$$



"Cyclotron Frequency"



Accelerating "DEES"

For a proton:

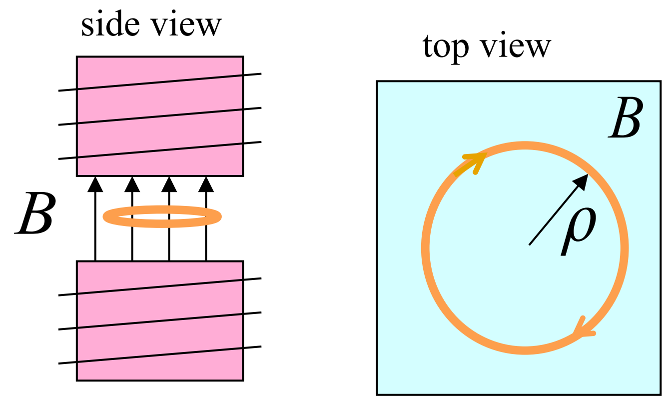
$$f_C = 15.2 \times B[T] \text{ MHz}$$

Understanding Beam Motion: Beam “rigidity”

- The relativistically correct form of Newton’s Laws for a particle in an electromagnetic field is:

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right); \quad \vec{p} = \gamma m \vec{v}$$

- A particle of unit charge in a uniform magnetic field will move in a circle of radius



$$\rho = \frac{p}{eB}$$

$$(B\rho) = \frac{p}{e}$$

constant for fixed energy!

T-m^2/s=V

$$(B\rho)c = \frac{pc}{e}$$

units of eV in our usual convention

Beam “rigidity” = constant at a given momentum (even when B=0!)

$$(B\rho)[\text{T-m}] = \frac{p[\text{eV/c}]}{c[\text{m/s}]} \approx \frac{p[\text{MeV/c}]}{300}$$

Remember forever!

If all magnetic fields are scaled with the momentum as particles accelerate, the trajectories remain the same
➔“synchrotron” [E. McMillan, 1945]

Example Beam Parameters

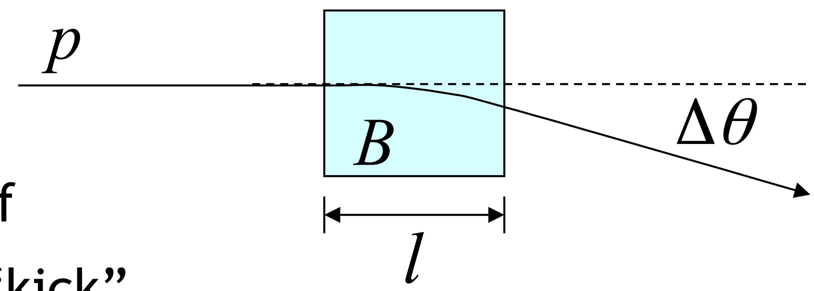
- Compare Fermilab LINAC (K=400 MeV) to LHC (K=7000 GeV)

Parameter	Symbol	Equation	Injection	Extraction
proton mass	m [GeV/c^2]		0.938	
kinetic energy	K [GeV]		.4	7000
total energy	E [GeV]	$K+mc^2$	1.3382	7000.938
momentum	p [GeV/c]	$\sqrt{E^2-(mc^2)^2}$	0.95426	7000.938
rel. beta	β	$(pc)/E$	0.713	0.999999991
rel. gamma	γ	$E/(mc^2)$	1.426	7461.5
beta-gamma	βγ	$(pc)/(mc^2)$	1.017	7461.5
rigidity	(Bρ) [T-m]	$p[\mathrm{GeV}]/(.2997)$	3.18	23353.

This would be the radius of curvature in a 1 T magnetic field *or* the field in Tesla needed to give a 1 m radius of curvature.

Thin lens approximation and magnetic "kick"

- If the path length through a transverse magnetic field is short compared to the bend radius of the particle, then we can think of the particle receiving a transverse "kick"

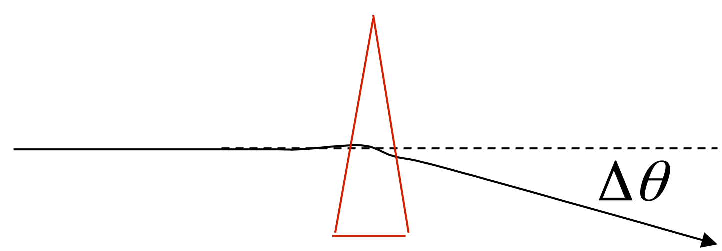


$$p_\perp \approx qvBt = qvB(l/v) = qBl$$

and it will be bent through small angle

$$\Delta\theta \approx \frac{p_\perp}{p} = \frac{Bl}{(B\rho)}$$

- In this "thin lens approximation", a dipole is the equivalent of a prism in classical optics.



Some Formalism (sorry)

- Define the "gradient" operator

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Field Multipole Expansion

- Formally, in a current free region, the curl of the magnetic field is:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = 0$$

- This means that the magnetic field can be expressed as the gradient of a scalar:

$$\vec{B} = -\overrightarrow{\nabla}\phi$$

- The zero divergence then gives us:

Laplace Equation

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) = 0$$

- If the field is *uniform* in z, then $\delta\phi/\delta z=0$, so

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

- The general solution is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \Rightarrow \varphi(x,y) = \mathrm{Re} \sum_{m=0}^{\infty} C_m (x+iy)^m$$

- Solving for B components

$$B_x = -\frac{\partial \varphi}{\partial x} = -\mathrm{Re} \sum_{m=1}^{\infty} mC_m (x+iy)^{m-1} = -\mathrm{Im} \sum_{m=1}^{\infty} imC_m (x+iy)^{m-1}$$

$$B_y = -\frac{\partial \varphi}{\partial y} = -\mathrm{Re} \sum_{m=1}^{\infty} imC_m (x+iy)^{m-1}$$

- Combining and redefining the constants

$$B_y + iB_x = \sum_{n=0}^{\infty} K_n (x+iy)^n ; K_n = i(n+1)C_{n+1}$$

Note order!

- We can express the complex numbers in notation

K_n is complex

r is real

$$B_y + iB_x = \sum_{n=0}^{\infty} K_n (x+iy)^n = \sum_{n=0}^{\infty} K_n r^n e^{in\theta}$$

$$= \sum_{n=0}^{\infty} |K_n| e^{i\delta_n} r^n e^{in\theta}$$

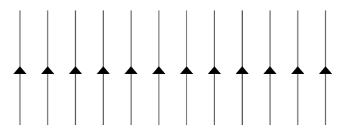
Amplitude

rotation

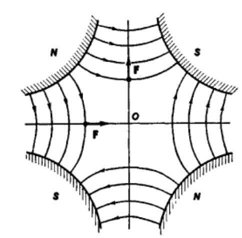
- In our general expression $$B_y + iB_x = \sum_{n=0}^{\infty} |K_n| e^{i\delta_n} r^n e^{in\theta}$$

the phase angle δ_m represents a rotation of each component about the z axis. Set all δ_m =0 for the moment, and we see the following symmetry properties for the first few multipoles

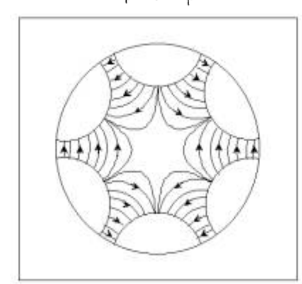
$$n = 0 \quad \Rightarrow \quad B_x = 0 \qquad ; B_y = |K_0| \qquad \equiv \text{dipole}$$



$$\begin{aligned} n = 1 \quad \Rightarrow \quad & B_x(r,0) = 0 && ; B_y(r,0) = r|K_1| && \equiv \text{quadrupole} \\ & B_x(r,\pi/2) = r|K_1| && ; B_y(r,\pi/2) = 0 \\ & B_{x,y}(r,\theta+\pi) && = -B_{x,y}(r,\theta) \end{aligned}$$



$$\begin{aligned} n = 2 \quad \Rightarrow \quad & B_x(r,0) = 0 && ; B_y(r,0) = r^2|K_2| && \equiv \text{sextupole} \\ & B_x(r,\pi/4) = r^2|K_2| && ; B_y(r,\pi/4) = 0 \\ & B_{x,y}(r,\theta+\pi/2) && = -B_{x,y}(r,\theta) \end{aligned}$$



- Back to Cartesian Coordinates. Expand by differentiating both sides *n* times wrt *x*

$$B_y + iB_x = \sum_{n=0}^{\infty} K_n (x+iy)^n$$

$$\Rightarrow \left[\left.\frac{\partial^n B_y}{\partial x^n}\right|_{x=y=0} + i\left.\frac{\partial^n B_x}{\partial x^n}\right|_{x=y=0} \right] = n! K_n$$

- And we can rewrite this as

$$B_y + iB_x = \sum_{n=0}^{\infty} \frac{1}{n!}\left(B_n + i\widetilde{B}_n\right)(x+iy)^n \quad ; B_n \equiv \left.\frac{\partial^n}{\partial x^n} B_y\right|_{x=y=0}$$

"normal"

$$\widetilde{B}_n \equiv \left.\frac{\partial^n}{\partial x^n} B_x\right|_{x=y=0}$$

"skew"

- "Normal" terms always have B_x=0 on *x* axis.
- "Skew" terms always have B_y=0 on *x* axis.
- Generally define

$$B' \equiv B_1, B'' \equiv B_2, \widetilde{B}' \equiv \widetilde{B}_1, \widetilde{B}'' \equiv \widetilde{B}_2, \text{etc}$$

- Expand first few terms…

$$B_y = B_0 + B'x - \tilde{B}'y + \frac{B''}{2}\left(x^2 - y^2\right) - \tilde{B}''xy + \ldots$$

$$B_x = \tilde{B}_0 + \tilde{B}'x + B'y + \frac{\tilde{B}''}{2}\left(x^2 - y^2\right) + B''xy + \ldots$$

dipole quadrupole sextupole

- Note: in the absence of skew terms, on the *x* axis

$$B_y = B_0 + B'x + \frac{B''}{2}x^2 + \frac{B'''}{6}x^3 \ldots + \frac{B_n}{n!}x^n$$

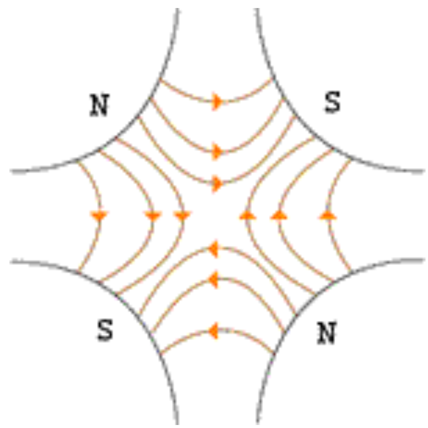
dipole quadrupole sextupole octupole

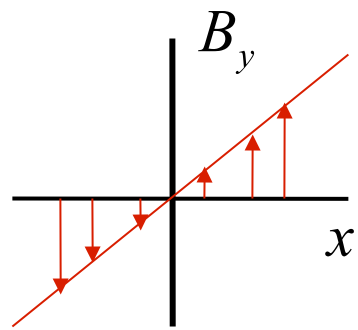
Application of Multipoles

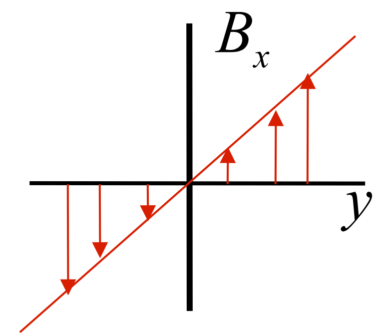
$$\vec{\nabla} \times \vec{B} = 0$$

$$\rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

- Dipoles: bend
- Quadrupoles: focus or defocus

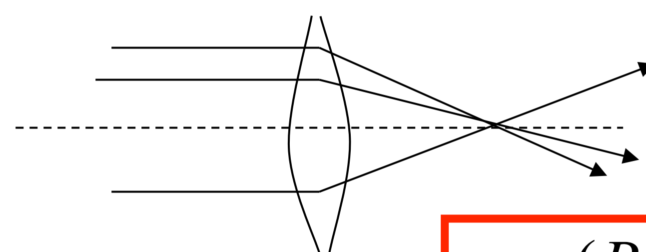






- A positive particle coming out of the page off center in the horizontal plane will experience a *restoring* kick

$$\Delta\theta \approx -\frac{B_x(x)l}{(B\rho)} = -\frac{B'lx}{(B\rho)}$$



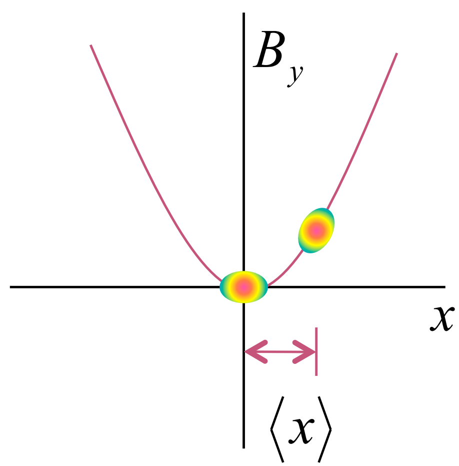
$$f = \frac{(B\rho)}{B'l}$$

Sextupoles

- Sextupole magnets have a field (on the principle axis) given by

$$B_y(x) = \frac{1}{2} B'' x^2$$

- One common application of this is to provide an effective position-dependent gradient.



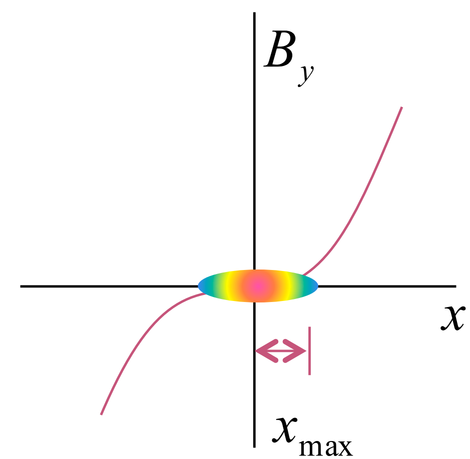
$$B'_{eff} = \langle x \rangle B''$$

Octupoles

- In a similar way, octupoles have a field given by

$$B_y(x) = \frac{1}{6} B''' x^3$$

- So high *amplitude* particles will see a different average gradiant



$$B'_{eff} = \frac{\langle x_{max}^2 \rangle}{2} B'''$$