## Off Momentum Particles

## Offr-Momentum Particles

- Our previous discussion implicitly assumed that all particles were at the same momentum
- Each quad has a constant focal length
- There is a single nominal trajectory
- In practice, this is never true. Particles will have a distribution about the nominal momentum
- We will characterize the behavior of off-momentum particles in the following ways
- "Dispersion" (D): the dependence of position on deviations from the nominal momentum

$$
\Delta x(s)=D_{x}(s) \frac{\Delta p}{p_{0}}
$$

$D$ has units of length
" "Chromaticity" $(\eta)$ : the change in the tune caused by the different focal lengths for off-momentum particles

$$
\Delta v_{x}=\xi_{x} \frac{\Delta p}{p_{0}} \quad\left(\text { sometimes } \frac{\Delta v_{x}}{v_{x}}=\xi_{x} \frac{\Delta p}{p_{0}}\right)
$$

- Path length changes (momentum compaction)

$$
\frac{\Delta L}{L}=\alpha \frac{\Delta p}{p}
$$

## Treating offrnomentum Particle Motion

- We have our original equations of motion

$$
\begin{aligned}
& x^{\prime \prime}=-\frac{B_{y}}{(B \rho)}\left(1+\frac{x}{\rho}\right)^{2}+\frac{\rho+x}{\rho^{2}} \\
& y^{\prime \prime}=\frac{B_{x}}{(B \rho)}\left(1+\frac{x}{\rho}\right)^{2}
\end{aligned}
$$

- Note that $\rho$ is still the nominal orbit. The momentum is the $(B \rho)$ term. If we leave ( $B \rho$ ) in the equation as the nominal value, then we must replace $(B \rho)$ with $(B \rho) p / p_{0}$, so

$$
\begin{aligned}
& x^{\prime \prime}=-\frac{B_{y}}{(B \rho)}\left(\frac{p_{0}}{p}\right)\left(1+\frac{x}{\rho}\right)^{2}+\frac{\rho+x}{\rho^{2}}=-\frac{B_{y}}{(B \rho)}\left(1-\frac{\Delta p}{p_{0}}\right)\left(1+\frac{x}{\rho}\right)^{2}+\frac{\rho+x}{\rho^{2}}=(\ldots)+\frac{B_{y}}{(B \rho)} \frac{\Delta p}{p_{0}} \\
& y^{\prime \prime}=\frac{B_{x}}{(B \rho)}\left(\frac{p_{0}}{p}\right)\left(1+\frac{x}{\rho}\right)^{2}=\frac{B_{x}}{(B \rho)}\left(1-\frac{\Delta p}{p_{0}}\right)\left(1+\frac{x}{\rho}\right)^{2}=(\ldots)-\frac{B_{x}}{(B \rho)} \frac{\Delta p}{p_{0}}
\end{aligned}
$$

Where we have kept only the lowest term in $\Delta p / p$. If we also keep only the lowest terms in $B$, then this term vanishes except in bend sectors, where

$$
\frac{B_{y}}{(B \rho)}=\frac{B_{0}}{(B \rho)}=\frac{1}{\rho} \Rightarrow x^{\prime \prime}+K(s) x=\frac{1}{\rho} \delta
$$

- This is a second order differential inhomogeneous differential equation, so the solution is

$$
x(s)=x_{0} C(s)+x_{0}^{\prime} S(s)+\delta d(s)
$$

$$
x^{\prime}(s)=x_{0} C^{\prime}(s)+x_{0}^{\prime} S^{\prime}(s)+\delta d^{\prime}(s)
$$

Where $\mathrm{d}(\mathrm{s})$ is the solution particular solution of the differential equation

$$
d^{\prime \prime}+K d=\frac{1}{\rho}
$$

- We solve this piecewise, for $K$ constant and find

$$
\begin{aligned}
& K>0: d(s)=\frac{1}{\rho K}(1-\cos \sqrt{K} s) \\
& d^{\prime}(s)=\frac{1}{\rho \sqrt{K}} \sin \sqrt{K} s \\
& K<0: d(s)=-\frac{1}{\rho K}(1-\cosh \sqrt{K} s) \\
& d^{\prime}(s)=\frac{1}{\rho \sqrt{K}} \sinh \sqrt{K} s
\end{aligned}
$$

- Recall

$$
x^{\prime \prime}+\left(\frac{1}{\rho^{2}}+B_{y}^{\prime}\right) x \Rightarrow K=\left(\frac{1}{\rho^{2}}+B_{y}^{\prime}\right)
$$

## Example: FODO Cell

- We look at our symmetric FODO cell, but assume that the drifts are bend magnets


For a thin lens $d \sim d^{\prime} \sim 0$. For a pure bend magnet
$K=\frac{1}{\rho^{2}}: \quad d(s)=\frac{1}{\rho K}(1-\cos \sqrt{K} s)=\rho\left(1-\cos \frac{s}{\rho}\right) \approx \frac{1}{2 \rho} s^{2} \rightarrow \frac{1}{2} \frac{L^{2}}{\rho}=\frac{1}{2} \theta L$

$$
d^{\prime}(s)=\frac{1}{\rho \sqrt{K}} \sin \sqrt{K} s \quad=\sin \frac{s}{\rho} \quad \approx \frac{s}{\rho} \rightarrow \theta
$$

Leading to the transfer Matrix
$M=\left(\begin{array}{ccc}1 & 0 & 0 \\ -\frac{1}{2 f} & 1 & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{lll}1 & L & \frac{L \theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{ccc}1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{ccc}1 & L & \frac{L \theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{ccc}1 & 0 & 0 \\ -\frac{1}{2 f} & 1 & 0 \\ 0 & 0 & 1\end{array}\right)=\left(\begin{array}{cc}1-\frac{L^{2}}{2 f^{2}} & 2 L\left(1+\frac{L}{2 f}\right) \\ 2 L \theta\left(1+\frac{L}{4 f}\right) \\ -\frac{L}{2 f^{2}}+\frac{L^{2}}{4 f^{3}} & 1-\frac{L^{2}}{2 f^{2}} \\ 0 & 2 \theta\left(1-\frac{L}{4 f}-\frac{L^{2}}{8 f^{2}}\right) \\ 0 & 1\end{array}\right)$

Solving for Lattice Functions

- Solve for
$\left(\begin{array}{c}D \\ D^{\prime} \\ 1\end{array}\right)=\mathbf{M}\left(\begin{array}{c}D \\ D^{\prime} \\ 1\end{array}\right)$
- As you'll solve in the homework

$$
\begin{aligned}
& \sin \frac{\mu}{2}=\frac{L}{2 f} \\
& \beta_{F, D}=\frac{2 L\left(1 \pm \sin \frac{\mu}{2}\right)}{\sin \mu} \\
& D_{F, D}=\frac{\theta L\left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{\sin ^{2} \frac{\mu}{2}} \\
& \alpha_{F, D}=D_{F, D}^{\prime}=0
\end{aligned}
$$

## Monentum Compaction and Slip Factor

- In general, particles with a high momentum will travel a longer path length. We have

$$
\begin{aligned}
C\left(p_{0}\right) & =\oint d s \\
C\left(p_{0}+\Delta p\right) & =\oint\left(1+\frac{D}{\rho} \frac{\Delta p}{p_{0}}\right) d s \\
\frac{\Delta C}{C} & =\frac{\oint \frac{D}{\rho} d s}{\oint d s} \delta=\left(\frac{D}{\rho}\right) \delta \\
& \equiv \alpha \delta
\end{aligned}
$$

- The slip factor is defined as the fractional
 change in the orbital period
- Note that
$=\alpha \frac{\Delta p}{p}-\frac{1}{\gamma^{2}} \frac{\Delta p}{p}$
$\gamma<\gamma_{T}: \quad \eta<0$ higher energy particles take less time to go around
$\gamma>\gamma_{T}: \quad \eta>0$ higher energy particles take more time to go around
$=\left(\frac{1}{\gamma_{T}^{2}}-\frac{1}{\gamma^{2}}\right) \frac{\Delta p}{p}$
$\gamma=\gamma_{T}: \quad \eta=0 \quad$ "transition"
$\equiv \eta \frac{\Delta p}{p}$


## Transitcion y

- In homework, you showed that for a simple FODO CELL

$$
\beta_{\text {max }, \text { min }}=2 L \frac{\left(1 \pm \sin \frac{\mu}{2}\right)}{\sin \mu} ; \text { and } D_{\text {max }, \text { min }}=\theta L \frac{\left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{\sin ^{2} \frac{\mu}{2}}
$$

- If we assume they vary $\sim$ linearly between maxima, then for small $\mu$
- It follows

$$
\begin{aligned}
\langle\beta\rangle & \approx \frac{2 L}{\mu} ;\langle D\rangle \approx \frac{4 \theta L}{\mu^{2}}=4 \frac{L^{2}}{\mu^{2} \rho}=\frac{\langle\beta\rangle^{2}}{\rho} \\
v & \approx \frac{R}{\langle\beta\rangle} \approx \frac{\rho}{\langle\beta\rangle} \Rightarrow\langle D\rangle \approx \frac{\rho}{v^{2}} \\
\alpha_{C} & =\frac{1}{C} \oint \frac{D}{\rho} d s \approx \frac{1}{\rho}\langle D\rangle \approx \frac{1}{v^{2}} \\
\gamma_{t} & =\frac{1}{\sqrt{\alpha_{C}}} \approx v
\end{aligned}
$$

- This approximation generally works better than it should


## Chromaticity

- In general, momentum changes will lead to a tune shift by changing the effective focal lengths of the magnets
- As we are passing trough a magnet, we can find the focal length as

$$
\begin{aligned}
\frac{1}{f} & =\frac{B^{\prime} l}{(B \rho)}=\frac{B^{\prime} l}{(B \rho)_{0}} \frac{p_{0}}{p} \approx \frac{1}{f_{0}}\left(1-\frac{\Delta p}{p_{0}}\right) \\
\Delta \frac{1}{f} & =-\frac{1}{f_{0}} \frac{\Delta p}{p_{0}} \\
& \Rightarrow \Delta v=-\frac{1}{4 \pi} \sum_{i} \beta_{i} \frac{1}{f_{i}} \frac{\Delta p}{p_{0}} \equiv \xi \frac{\Delta p}{p_{0}}
\end{aligned}
$$

$$
\frac{1}{f_{0}}=\int_{0}^{L} \frac{B^{\prime}}{(B \rho)} d s
$$

- But remember that our general equation of motion is

$$
x^{\prime \prime}+\left(\frac{1}{\rho^{2}}+\frac{B^{\prime}(s)}{(B \rho)}\right) x=0 \equiv x^{\prime \prime}+K(s) x
$$

- Clearly, a change in momentum will have the same effect on the entire focusing term, so we can write in general

$$
\xi=-\frac{1}{4 \pi} \oint \beta_{0}(s) K(s) d s
$$

## Review: Closed Form Solution

○ Our linear equations of motion are in the form of a "Hill's Equation"
$x^{\prime \prime}+K(s) x=0 ; \quad K(s+C)=K(s) \longleftarrow$ Consider only periodic systems at the moment

- If $K$ is a constant $>0$, then $x(s)=A \cos (\sqrt{K} s+\delta)$ so try a solution of the form
$x(s)=A w(s) \cos (\psi(s)+\delta)$
assume $w(s+C)=w(s)$, BUT
$\psi(s+C) \neq \psi(s)$
- If we plug this into the equation, we get
$x^{\prime \prime}+K x=A\left(w^{\prime \prime}-w \psi^{\prime 2}+K w\right) \cos (\psi+\delta)-A\left(2 w^{\prime} \psi^{\prime}+w \psi^{\prime \prime}\right) \sin (\psi+\delta)=0$
- Coefficients must independently vanish, so the sin term gives
$2 w^{\prime} \psi^{\prime}+w \psi^{\prime \prime}=0 \underset{\text { mutliply by } w}{ } 2 w w^{\prime} \psi^{\prime}+w^{2} \psi^{\prime \prime}=\left(w^{2} \psi^{\prime}\right)^{\prime}=0 \Rightarrow \psi^{\prime}=\frac{k}{w^{2}}$
- If we re-express our general solution
$x=w\left(A_{1} \cos \psi+A_{2} \sin \psi\right)$
$x^{\prime}=\left(A_{1} w^{\prime}+A_{2} w \psi^{\prime}\right) \cos \psi+\left(A_{2} w^{\prime}-A_{1} w \psi^{\prime}\right) \sin \psi$
$=\left(A_{1} w^{\prime}+A_{2} \frac{k}{w}\right) \cos \psi+\left(A_{2} w^{\prime}-A_{1} \frac{k}{w}\right) \sin \psi$
We'll see this much later
$w^{\prime \prime}-\frac{k^{2}}{w^{3}}+K w=0$


## Chronaticity in Terns of Laitice Functions

- On the last page, we derived two relationships when solving our Hill's equation

$$
\begin{aligned}
\psi^{\prime} & =\frac{k}{w^{2}(s)} \\
w^{\prime \prime}(s)+K(s) w(s)-\frac{k}{w^{3}(s)} & =0 \Rightarrow(\sqrt{\beta})^{\prime \prime}+K \sqrt{\beta}-\frac{1}{\beta^{3 / 2}}=0 \\
(\sqrt{\beta})^{\prime} & =\frac{1}{2} \frac{1}{\sqrt{\beta}} \beta^{\prime}=-\frac{\alpha}{\sqrt{\beta}} \\
(\sqrt{\beta})^{\prime \prime} & =-\frac{\alpha^{\prime}}{\sqrt{\beta}}+\frac{1}{2} \frac{\alpha}{\beta^{3 / 2}} \beta^{\prime}=-\frac{\alpha^{\prime}}{\sqrt{\beta}}-\frac{\alpha^{2}}{\beta^{3 / 2}} \quad \begin{array}{l}
\text { Multiply } \\
\text { by } \mathrm{B}^{3 / 2}
\end{array} \\
& \Rightarrow K \beta^{2}-\beta \alpha^{\prime}-\alpha^{2}=1
\end{aligned}
$$

- (We're going to use that in a few lectures), but for now, divide by $B$ to get

$$
K \beta=\frac{1+\alpha^{2}}{\beta}+\alpha^{\prime}=\frac{\beta \gamma}{\beta}+\alpha^{\prime}=\gamma+\alpha^{\prime}
$$

- So our general expression for chromaticity becomes

$$
\xi=-\frac{1}{4 \pi} \oint\left(\gamma(s)+\alpha^{\prime}(s)\right) d s
$$

## Chromaticity and Sextupoles

- we can write the field of a sextupole magnet as

$$
B(x)=\frac{1}{2} B^{\prime \prime} x^{2} \quad\left(\text { often expressed } b_{2} x^{2}\right)
$$

- If we put a sextupole in a dispersive region then off momentum particles will see a gradient

$$
B^{\prime}(x=D \delta) \approx B^{\prime \prime} D \frac{\Delta p}{p_{0}}
$$ which is effectively like a position $p_{0}$ dependent quadrupole, with a focal length given by



$$
\frac{1}{f_{e f f}}=\frac{B^{\prime \prime}}{(B \rho)} L D \frac{\Delta p}{p_{0}}
$$

- So we write down the tune-shift as

$$
\begin{aligned}
\Delta v & =\frac{1}{4 \pi} \beta \frac{1}{f_{\text {eff }}}=\frac{1}{4 \pi} \frac{\beta B^{\prime \prime}}{(B \rho)} L D \frac{\Delta p}{p_{0}} \equiv \xi \frac{\Delta p}{p_{0}} \\
& \Rightarrow \xi_{s}=\frac{1}{4 \pi} \frac{\beta B^{\prime \prime}}{(B \rho)} L D
\end{aligned}
$$

- Note, this is only valid when the motion due to mometum is large compared to the particle spread (homework problem)

