Hamiltonian systems

$H[\bar{q}, \bar{p}; t]$:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ -\dot{p}_i = \dfrac{\partial H}{\partial q_i} \end{cases} \qquad \begin{matrix} i = 1, 2, \dots, n \\ q_i(t_0) = q_i^0 \ , \ p_i(t_0) = p_i^0 \end{matrix}$$

q_i - generalized coordinates

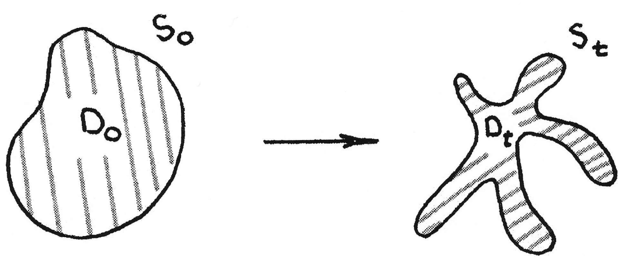
p_i - generalized momentums

n - number of degrees of freedom

$(q_i, p_i) \equiv (q, p)$ - canonical variables, which form phase space

Properties of the phase space

1) Phase space trajectories never cross each other
2) Phase space volume is conserved (Liouvile Theorem).
3) $S_0 \to S_t$



$$\int_{D_0} d\bar{q}_0 \, d\bar{p}_0 = \int_{D_t} d\bar{q} \, d\bar{p}$$

i.e. phase space is incompressible

In order to describe behavior of the system it is useful to know integrals of motion:

$f(\bar{q}, \bar{p}) = \text{const} \quad \forall t$

Problem 1. Show that for autonomus system, the Hamiltonian is an integral of motion.
(i.e. that phase space trajectory lies on a (2n-1)-D hyper surface)

Hamiltonian systems allow canonical transformations, which are preserving form of equations of motion:

$(\bar{q}, \bar{p}) \to (\bar{q}', \bar{p}') : \ H(\bar{q}, \bar{p}) \to H'(\bar{q}', \bar{p}') \ : \ \dot{q}_i' = \dfrac{\partial H'}{\partial p_i'} \ \& \ \dot{p}_i' = -\dfrac{\partial H'}{\partial q_i'}$

Action angle variables

Consider a canonical transformation of the form:

$(\bar{q}, \bar{p}) \to (\bar{\alpha}, \bar{J})$:

$$H(\bar{q}, \bar{p}) \to H(\bar{J}) \quad \curvearrowright \quad \begin{cases} \dot{\alpha}_i = \dfrac{\partial H'}{\partial J_i} \\ \dot{J}_i = \dfrac{-\partial H'}{\partial \alpha_i} = 0 \end{cases} \curvearrowright \quad \begin{aligned} &\bar{\alpha} = \bar{\alpha}_0 + \bar{\omega} t, \quad \omega_i = \frac{\partial H'}{\partial J_i} \\ &\bar{J} = \bar{J}_0 = \text{const} \end{aligned}$$

If $\exists$ canonical transformation such a $(\bar{q}, \bar{p}) \to (\bar{\alpha}, \bar{J})$, then system is called to be integrable.

$\boxed{n=1}$

$$H[q,p] = \frac{p^2}{2m} + U(q)$$

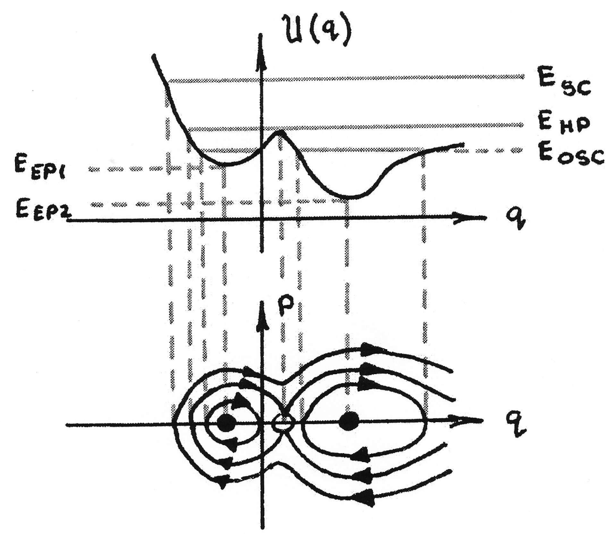
Using the first integral of motion:

$$E = \frac{p^2}{2m} + U(q) = \text{const}$$

$$\curvearrowright \quad \frac{dq}{dt} = \sqrt{2m}\,\sqrt{E - U(q)}$$

$$(t - t_0) = \sqrt{2m}^{-1} \int_{q_0}^{q} \frac{d\tilde{q}}{\sqrt{E - U(\tilde{q})}}$$

E, t_0 are const

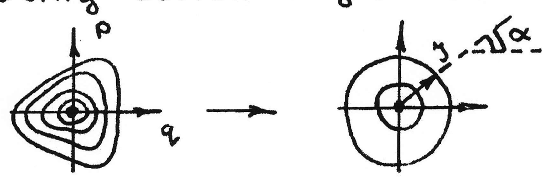


Motion is possible only for $U(q) < E$

$q_\pm$: $U(q) = E$ - stop points

q : $\partial_q U(q) = 0$ - elliptic/hyperbolic stationary points

Using action-angle variables (which are always exist for $n=1$)



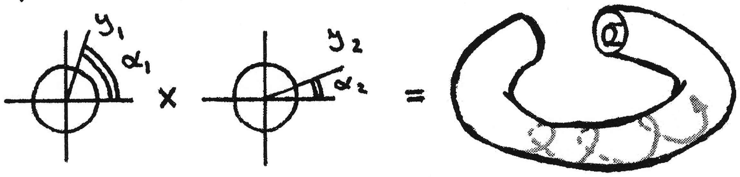
$$J = \frac{1}{2\pi} \oint p\, dq$$

Problem 2. Consider a pendulum $H = \dfrac{p^2}{2ml^2} - mgl\cos\varphi$

- show that equation of motion is $\ddot{\varphi} + \omega_0^2 \sin\varphi = 0$
- describe the phase space
- How u can describe the dynamics around $\varphi = 0, \pi$
- Find period of oscillations and action-angle variables

$\boxed{n=2}$

If it is possible to perform canonical transformation to action-angle variables $(q_1, q_2, p_1, p_2) \to (\alpha_1, \alpha_2, J_1, J_2)$, the phase-space is fibrated of toruses:



Phase space of the integrable system with 2 degrees of freedom

In general, from torus to torus, one will have different value of

$$\frac{\omega_1}{\omega_2} = \frac{\omega_1(J_1, J_2)}{\omega_2(J_1, J_2)}$$

- commensurate frequencies, i.e. $\omega_1/\omega_2 = k/m$, $k, m \in \mathbb{N}$:

  $\alpha_1(t) = \omega_1 t + \alpha_1^0 \Rightarrow \alpha_1(t+T) = \omega_1 t + \alpha_1^0 + 4\pi k$

  $\alpha_2(t) = \omega_2 t + \alpha_2^0 \Rightarrow \alpha_2(t+T) = \omega_2 t + \alpha_2^0 + 4\pi m$

  where $T = 2\pi\left(\frac{k}{\omega_1} + \frac{m}{\omega_2}\right)$, which means the motion is periodic.

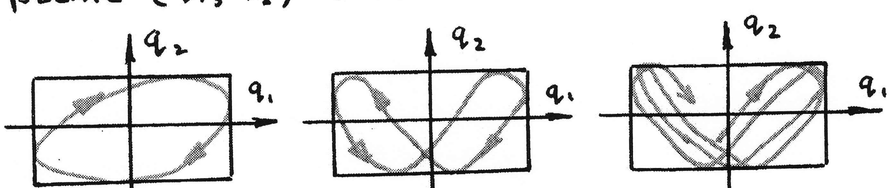
- incommensurate frequencies, i.e. $\omega_1/\omega_2 \notin \mathbb{Q}$, $\in \mathbb{I} = \mathbb{R}/\mathbb{Q}$:

  the motion is quasi-periodic and phase space trajectory covering the whole surface of torus.

The n-D torus $(n \geq 2)$ is called resonance torus if

$$\sum_{i=1}^{n} k_i \omega_i(J_1, \dots, J_n) = 0 \qquad \text{where} \quad k_i \in \mathbb{Q}, \quad i = 1, 2, \dots, n.$$

Projection of the phase space trajectory onto coordinate plane (q_1, q_2) is called Lissajous curves



$\omega_1/\omega_2 = 1$ $\qquad$ $\omega_1/\omega_2 = 1/2$ $\qquad$ $\omega_1/\omega_2 \approx 1/2$ But $\notin \mathbb{Q}$

Problem 3. Consider a 2-D harmonic oscillator

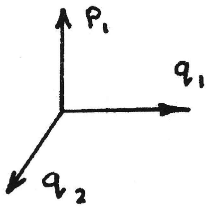
$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2).$$

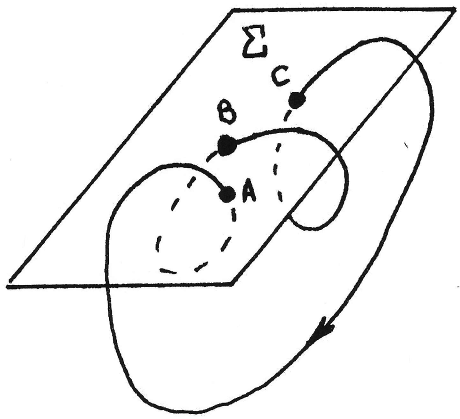
Find canonical transformation to action-angle variables & Expression for $H(J_1, J_2)$. Draw phase space torus, and show where is $\alpha_1, \alpha_2, J_1, J_2$.

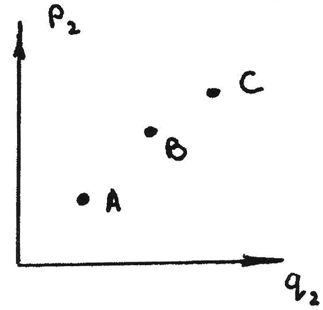
Poincare map

Consider a Hamiltonian system with 2 degrees of freedom. The motion is constrained on the 3-D hypersurface in 4-D phase space:

$$H(\bar{q}, \bar{p}) = E \quad \rightsquigarrow \quad p_2 = p_2(q_1, q_2, p_1) = p_2(q_1, q_2, p_1, E)$$



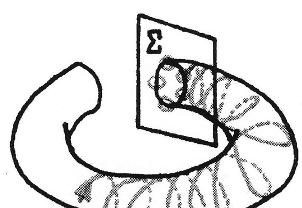


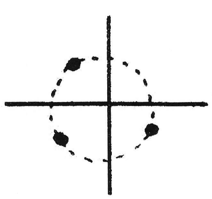


Then we can fix a surfase Σ, and study intersection of it and phase space trajectory:

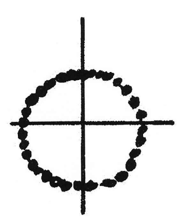
$$p_2(q_1, q_2, p_1)\Big|_{\substack{\Sigma:\ q_2 = \text{const} \\ q_1(t_i^* + \delta t) - q_1(t_i^*) > 0}}$$

which gives us a map $\quad A \rightarrow B = \Phi(A) \rightarrow C = \Phi(B) = \Phi \circ \Phi(A)$

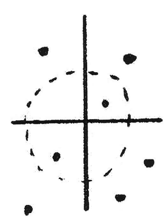




periodic



quasi-per.



chaotic

Anharmonicity

Consider a non-linear oscillator with a Hamiltonian:

$$H = \frac{p^2}{2} + \frac{\omega_0^2}{2} q^2 + \frac{\alpha}{3} q^3 + \frac{\beta}{4} q^4$$

$$\begin{cases} \dfrac{\partial H}{\partial p} = \dot{q} \\ -\dfrac{\partial H}{\partial q} = \dot{p} \end{cases} \rightarrow -\alpha q^2 - \beta q^3 = \ddot{q} + \omega_0^2 q \quad (1)$$

We will look for solution in a form:

$$q = q^{(1)} + q^{(2)} + q^{(3)} + \dots$$

where $q^{(1)} = a \cos \omega t$ with $\omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + \dots$

Rewriting (1) as

$$\frac{\omega_0^2}{\omega^2} \ddot{q} + \omega_0^2 q = -\alpha q^2 - \beta q^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{q}$$

- For $q = q^{(1)} + q^{(2)}$ and $\omega = \omega_0 + \omega^{(1)}$, and keeping terms up to a second order of smallness:

$$\ddot{q}^{(2)} + \omega_0^2 q^{(2)} = -\alpha a^2 \cos^2 \omega t + 2\omega_0 \omega^{(1)} a \cos \omega t =$$

$$= -\frac{\alpha a^2}{2} - \frac{\alpha a^2}{2} \cos 2\omega t + \underbrace{2\omega_0 \omega^{(1)} a \cos \omega t}_{\text{resonant condition}}$$

$$\rightarrow \omega^{(1)} = 0 \quad \rightarrow \quad q^{(2)} = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2} \cos 2\omega t$$

- For $q = q^{(1)} + q^{(2)} + q^{(3)}$ and $\omega = \omega_0 + \omega^{(2)}$ one have

$$\ddot{q}^{(3)} + \omega_0^2 q^{(3)} = -2\alpha q^{(1)} q^{(2)} - \beta q^{(1)3} + 2\omega_0 \omega^{(2)} q^{(1)}$$

$$\rightarrow \ddot{q}^{(3)} + \omega_0^2 q^{(3)} = -a^3 \left[\frac{\beta}{4} + \frac{\alpha^2}{6\omega_0^2}\right] \cos 3\omega t + \underbrace{a\left[2\omega_0 \omega^{(2)} + \frac{5a^2\alpha^2}{6\omega_0^2} - \frac{3}{4}\alpha^2 \beta\right] \cos \omega t}_{\text{resonant condition}}$$

$$\begin{cases} \omega^{(2)} = \left(\dfrac{3\beta}{8\omega_0} - \dfrac{5\alpha^2}{12\omega_0^3}\right) a^2 \propto a^2 \\ q^{(3)} = \dfrac{a^3}{16\omega_0^2} \left(\dfrac{\alpha^2}{3\omega_0^2} + \dfrac{\beta}{2}\right) \cos 3\omega t \end{cases}$$

Sextupoles

Equations of motion:

$$\begin{cases} x'' + K_x(s)\,x = \frac{1}{2}\, m(s)\,(x^2 - z^2) \\ z'' + K_z(s)\,z = -\,m(s)\,x z \end{cases} \qquad m(s) = \frac{S(s)}{B\rho} = \frac{1}{B\rho}\,\frac{d^2 B_z}{dx^2}\bigg|_{x=z=0}$$

Unperturbed motion $q(s) = \sqrt{2J\beta}\,\cos(\phi(s) + \varphi_0)$, where:

$$\begin{cases} 2\beta\beta'' - \beta'^2 + 4\beta^2 K(s) = 4 \\ \phi(s) = \int_0^s \frac{ds}{\beta(s)} \end{cases} \qquad \nu = \frac{1}{2\pi} \int_s^{s+C} \frac{ds}{\beta(s)}$$

<u>Floquet transform:</u>

$$q(s) = \sqrt{2J\beta}\,\cos\Psi(s) \;\rightarrow\; \xi = q/\sqrt{\beta} = \sqrt{2J}\,\cos\Psi(s)$$

$$\frac{d\xi}{d\Psi} = \frac{\alpha q + \beta q'}{\sqrt{\beta}} = -\sqrt{2J}\,\sin\Psi(s)$$

$$2J = \xi^2 + \left(\frac{d\xi}{d\Psi}\right)^2 \;\rightarrow\; \frac{d^2\xi}{d\Psi^2} + \nu^2 \xi = 0$$

Therefore

$$x'' + K_x(s)\,x = f(x,s) \;\rightarrow\; \underline{\frac{d^2\xi}{d\phi^2} + \nu^2\xi = \nu^2 \beta^{3/2} f(x,s)}$$

1D case

If $z \equiv 0$, $K_x(s) = \text{const}$, $m(s) = \text{const}$

$$\frac{d^2\xi}{d\Psi^2} + \nu_0^2\,\xi = -\frac{1}{2}\,\nu_0^2\, m\, \beta^{5/2}\, \xi^2$$

$x = \xi$ and $t = \Psi$, $\omega = \nu$, $\alpha = -\frac{1}{2} m \beta^{3/2}$ gives

$$x'' + \omega_0^2 x = \varepsilon\,\alpha\,\omega_0^2 x^2 \qquad x_0 = A\cos\omega_0 t$$

$$\begin{cases} x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots \\ \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots \end{cases} \;\rightarrow\; x_1'' + x_1 = \cancel{2A\frac{\omega_1}{\omega_0}\cos\omega t} + \frac{\alpha A^2}{2}(1 + \cos 2\omega t)$$

$$x_1 = \frac{1}{6}\,\alpha A^2\,(3 - 2\cos\omega t - \cos 2\omega t)$$

Next order gives $x_2'' + x_2 = 2\alpha x_0 x_1 - 2x_0''\frac{\omega_2}{\omega_0} \rightarrow \omega_2 = -\frac{5}{12}\alpha^2\omega_0 A^2$

$$\underline{\Delta\nu = -\frac{5}{48}\, m\, A_x^2\, \beta_x^4\, \nu_0}$$

time dependence

$x'' + \omega_0^2 x = g(t)$, where one can perform Fourier expansion of R.H.S.:

$$g(t) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos n\Omega t$$

$$\to x_n(t) = \frac{a_n}{\omega_0^2 - n^2\Omega^2} \cos n\Omega t \qquad \omega_0 \approx \pm n\Omega \text{ - resonance}$$

Hamiltonian formalism

$$H = \underbrace{\frac{1}{2}(p_x^2 + p_z^2) + \frac{1}{2}(K_x x^2 + K_z z^2)}_{H_0} + \underbrace{\frac{1}{6} m (x^3 - 3xz^2)}_{\epsilon H_1}$$

Unperturbed motion (asimuthally symmetric lattice)

$$q = \sqrt{2J\beta} \cos(\nu\theta)$$

$$p = -\sqrt{\frac{2J}{\beta}} \sin(\nu\theta)$$

$$J = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi} 2J \int_0^{2\pi} d\left(\nu \frac{s}{R}\right) \sin^2\left(\nu \frac{s}{R}\right) = J$$

$$\to H_0 = \frac{J}{\beta(s)}$$

In general one need to use generating function

$$F_1 = -\frac{\alpha + tg\,\psi}{2\beta} x^2$$

$$\begin{cases} x' = \dfrac{\partial F_1}{\partial x} \\ J = \dfrac{\partial F_1}{\partial \psi} \end{cases} \qquad \bar{H} = H + \frac{\partial F_1}{\partial s}$$

Consider

$$\int \frac{ds}{\beta} - \nu \frac{s}{c} = \int \frac{ds}{\beta} - \nu\theta \quad \to \quad \text{one can use } \varphi = \psi - \int \frac{ds}{\beta} + \nu\theta$$

$$F_2 = \bar{J}\left(\psi - \int \frac{ds}{\beta} + \nu\theta\right)$$

$$J = \partial F_2 / \partial \psi \qquad \varphi = \partial F_2 / \partial \bar{J}$$

$$H_0 = \frac{\nu}{R} \bar{J}$$

Canonical Perturbation theory

1D

$$H(y,\varphi) = H_0(y) + \epsilon H_1(y,\varphi,\theta)$$

Unperturbed motion $y_0 = \text{const}$, $\varphi = \omega\theta + \alpha$, $\omega = \partial H_0/\partial y$

We will look for $(y,\varphi) \to (\bar{y},\bar{\varphi})$:

$$\bar{H}(\bar{y},\bar{\varphi}) = \bar{H}_0(\bar{y}) + \epsilon \bar{H}_1(\bar{y}) + \epsilon^2 \bar{H}_2(\bar{y},\bar{\varphi},\theta)$$

with help of $F_2(\varphi,\bar{y},\theta)$:

$$\begin{cases} y = \partial F_2/\partial\varphi \\ \bar{\varphi} = \partial F_2/\partial\bar{y} \\ \bar{H} = H + \partial F_2/\partial\theta \end{cases}$$

where $F_2 = \underbrace{F_{20}}_{\text{trivial}} + \epsilon F_{21} + \epsilon^2 F_{22} + \dots$ $\quad F_{20} = \varphi\bar{y}$

$$y(\bar{y},\bar{\varphi}) = \bar{y} + \epsilon\frac{\partial F_{21}}{\partial\varphi} + \epsilon^2\frac{\partial F_{22}}{\partial\varphi} + \dots = \bar{y} + \Delta y$$

$$\varphi(\bar{y},\bar{\varphi}) = \bar{\varphi} - \epsilon\frac{\partial F_{21}}{\partial\bar{y}} - \epsilon^2\frac{\partial F_{22}}{\partial y} - \dots = \bar{\varphi} + \Delta\varphi$$

Thus keeping only first term of smallness:

- $H_0(y) = H_0(\bar{y} + \Delta y) = H_0(\bar{y}) + \Delta y \frac{\partial H_0}{\partial y} + \dots$

  where the use of $\omega_0 = \partial H_0/\partial y$ gives

  $$H_0(\bar{y},\bar{\varphi}) = H_0(\bar{y}) + \epsilon\omega_0\frac{\partial F_{21}}{\partial\varphi} + \dots$$

- $\epsilon H_1(y,\varphi,\theta) = \epsilon H_1(\bar{y},\bar{\varphi},\theta) + \dots$

$$\bar{H}(\bar{y},\bar{\varphi},\theta) = \underbrace{H_0(\bar{y})}_{\bar{H}_0} + \epsilon\underbrace{\left[H_1(\bar{y},\bar{\varphi},\theta) + \left(\omega_0\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta}\right)F_{21}\right]}_{\bar{H}_1} + \dots$$

Then we can treat $\bar{H}_1$ as $\bar{H}_1 = \underbrace{\langle\bar{H}_1\rangle}_{\text{time independent term}} + \{\bar{H}_1\}$

time independent term:

$$\langle\bar{H}_1\rangle = \frac{1}{(2\pi)^2}\oint \bar{H}_1(\bar{y},\bar{\varphi},\theta)\,d\bar{\varphi}\,d\theta$$

$$\Rightarrow \omega_0\frac{\partial F_{21}}{\partial\varphi} + \frac{\partial F_{21}}{\partial\theta} = -\{\bar{H}_1(\bar{y},\bar{\varphi},\theta)\} \quad (*)$$

gives

$$\bar{H}_1(\bar{y}) = H_0(\bar{y}) + \epsilon\langle H_1(\bar{y},\bar{\varphi},\theta)\rangle$$

and $\omega_1(\bar{J}) = \frac{\partial}{\partial \bar{J}} \langle H_1 \rangle$.

New phase space trajectories are given by

$$J(\bar{J}, \varphi) \approx \bar{J} + \epsilon \frac{\partial F_{21}}{\partial \varphi}$$

Periodicity gives

$$F_{21} = \sum_{m=-\infty}^{\infty} f_m(\bar{J}, \bar{\theta}) e^{im\bar{\varphi}} \qquad \text{and} \qquad \{H_1\} = \sum_{m=-\infty}^{\infty} h_m(\bar{J}, \theta) e^{im\bar{\varphi}}$$

substitution of these eqns. into (*) gives amplitudes of harmonics:

$$[im\omega_0 + \partial/\partial\theta] f_m = -h_m \quad \rightarrow \quad \underline{f_m = \frac{i}{2\sin(\pi m\omega_0)} \int_{\theta}^{\theta+2\pi} h_m(\bar{J}, \theta') e^{im\omega(\theta'-\theta-\pi)} d\theta'}$$

One can choose another way:

$$F_{21} = \sum_{m,n=-\infty}^{\infty} f_{m,n}(\bar{J}) e^{i(m\bar{\varphi} - n\bar{\theta})} \qquad \{H_1\} = \sum_{m,n=-\infty}^{\infty} h_{m,n}(\bar{J}) e^{i(m\bar{\varphi} - n\bar{\theta})}$$

$$\rightarrow \underline{F_{21} = i \sum_{m,n} \frac{h_{m,n}(\bar{J})}{m\omega_0 - n} e^{i(m\bar{\varphi} - n\theta)}}$$

<u>Example:</u>

$$H = H_0 + \epsilon H_1 = \left(\tfrac{1}{2} P_x^2 + \tfrac{1}{2} K_x(s) x^2\right) + \tfrac{1}{2} \Delta K_x(s) x^2 \qquad (\epsilon = \Delta K_x(s))$$

$$H = \frac{J_x}{\beta_x(s)} + \tfrac{1}{2} J_x \Delta K_x(s) \beta(s) [1 + \cos 2\varphi_x]$$

new Hamiltonian can be obtained using averaging:

$$\bar{H} = \frac{\bar{J}_x}{\beta_x(s)} + \tfrac{1}{2} \bar{J}_x \Delta K_x(s) \beta_x(s)$$

$$\bar{\varphi}(s) = \int \frac{\partial H}{\partial \bar{J}} ds = \int_0^s \frac{ds}{\beta_x(s)} + \frac{1}{2} \int_0^s \Delta K_x(s) \beta_x(s) ds$$

$$\Rightarrow \Delta\nu_x = \frac{1}{4\pi} \int \Delta K_x(s) \beta_x(s) ds$$

Only one Harmonic term $\epsilon H_1 = h_2 \cos 2\varphi = \frac{1}{2} \bar{J}_x \Delta K_x(s) \beta_x(s) \cos 2\varphi$

$$F_{21}(s) = -\frac{\bar{J}_x}{4\sin(2\pi\nu_x)} \int_s^{s+C} \Delta K_x(s') \beta_x(s') \sin 2[\bar{\varphi}_x - \psi_x(s') - \psi_x(s) - \pi\nu_x] ds'$$

$$J_x(\varphi_x, s) = \bar{J}_x + \frac{\partial F_{21}(\bar{J}_x, \varphi_x, s)}{\partial \varphi_x} \qquad \text{with } \bar{\varphi}_x \approx \varphi_x$$

Substitution of F_{21} gives

$$\eta_x(s) = \bar{\eta}_x \left\{ 1 - \frac{1}{2\sin(2\pi\nu_x)} \int \dots \right\}$$

$$\frac{\Delta \eta_x(s)}{\bar{\eta}_x} = - \frac{1}{2\sin(2\pi\nu_x)} \int \dots \quad = - \frac{\Delta\beta_x}{\beta_x}$$

$\boxed{\text{Resonance perturbation theory}}$

$$F_{21} = i \sum_{m,n} \frac{h_{mn}(\bar{\eta})}{m\omega_0 - n} e^{i(m\bar{\varphi} - n\theta)}$$

for 2 dimensions denominator becomes $m_x \nu_x + m_z \nu_z - n$

$$H = H_0 + \epsilon \bar{H}_1 + \epsilon \{\bar{H}_1\}$$

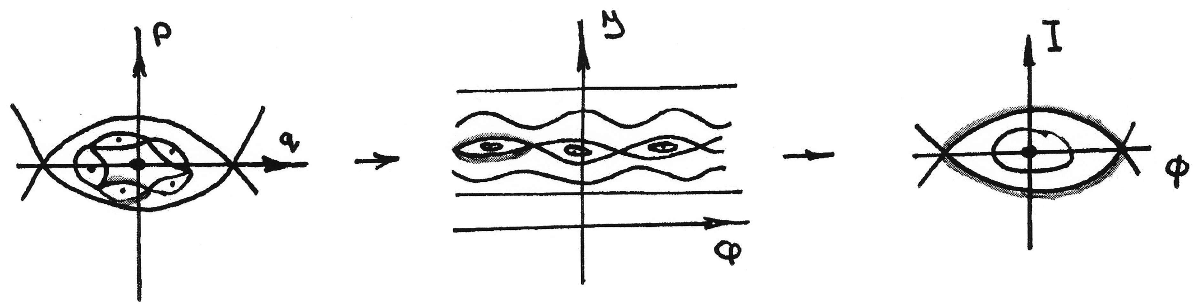
for $\nu \approx n/m$ $\quad A_{mn} \cos(m\varphi - n\theta) \approx \text{const}$

$$\Rightarrow H \approx \nu\eta + \underbrace{\alpha(\eta)}_{\bar{H}_1(\eta)} + \underbrace{f(\eta)}_{A_{mn}(\eta)} \cos(m\varphi - n\theta)$$

Then using $(\eta, \varphi) \to (I, \phi)$: $F_2(I, \varphi) = (\varphi - \frac{n}{m}\theta) \cdot I$

$$\begin{cases} \eta = \partial F_2 / \partial\varphi = I \\ \phi = \partial F_2 / \partial I = \varphi - \frac{n}{m}\theta \end{cases} \quad \to \quad \bar{H} = H + \frac{\partial F_2}{\partial \theta} = H - \frac{n}{m} I = \delta I + \alpha(I) + f(I)\cos m\phi$$

where $\delta = \nu - n/m$



1D Sextupole resonance

$H_1 = \frac{1}{6} m(s) x^3 \Rightarrow$ using $x = \sqrt{2J\beta}\cos\varphi$ and $\cos^3\varphi = \frac{1}{4}(\cos 3\varphi + 3\cos\varphi)$

$H_1 = \frac{1}{6\sqrt{2}} J^{3/2} \beta^{3/2}(s)\, m(s)\, (\cos 3\varphi + 3\cos\varphi)$

- Resonances: $\nu = n/3$ & $\nu = n$
- Averaging of $\langle H_1 \rangle_\varphi = 0$ gives $\Delta\nu = 0$

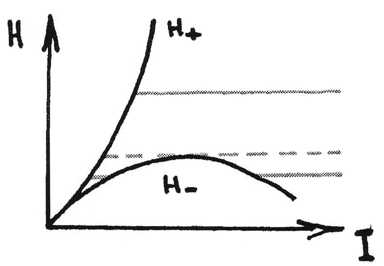
Resonance $3\nu = n$

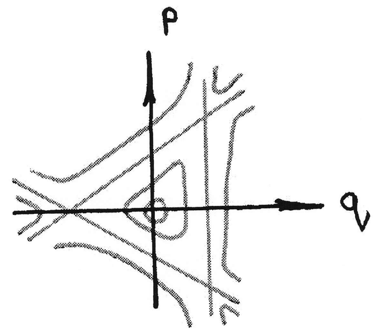
Let us to expand H_1 with respect to $\theta = s/R$ and keeping only resonance term $A_n = 2^{3/2} A_{3n}$

$H \approx \nu \cdot J + J^{3/2} A_n \cos(3\varphi - n\theta)$

$H = \delta I + I^{3/2} A_n \cos 3\phi$

where $\delta = \nu - n/3$ and $\delta / A_n > 0$





Separatrix of motion gives 3 unstable equilibrium points:

$\frac{\partial H}{\partial \phi} = 3\sin 3\phi = 0 \qquad \frac{\partial H}{\partial I} = \delta + \frac{3}{2} A_n \cdot I^{1/2} \cos 3\phi = 0$

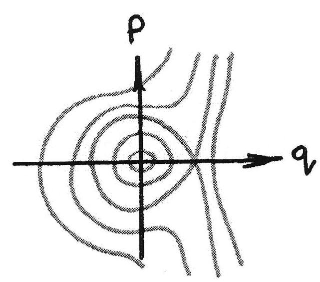
$I^{1/2} = -2\delta / 3A_n \cos 3\phi \rightarrow \{\pi/3, \pi, 5\pi/3\}$

$I^{1/2} = 2\delta / 3A_n$ ~ dynamical aperture

$A_x = \sqrt{2\beta I} = 2\delta\sqrt{2\beta} / 3A_n$

Resonance $\nu = n$

Only one Hyperbolic point $\phi = 0$ $(\phi = \pi)$



4. Nonlinear Resonance Problem

Using the resonance perturbation theory discussed in the class, describe the behavior around the resonance $3\nu = n$ for the lattice with sextupole and qubic nonlinearity:

$$H = \delta I + \alpha_0 I^2 + A_n I^{3/2} \cos 3\phi.$$

Consider cases $\alpha_0 > 0$ and $\alpha_0 < 0$. What you can say about the behavior when detuning from resonance $\delta = 0$.
(For simplisity assume that $\delta > 0$ and $A_n > 0$).