USPAS Accelerator Physics Final Exam

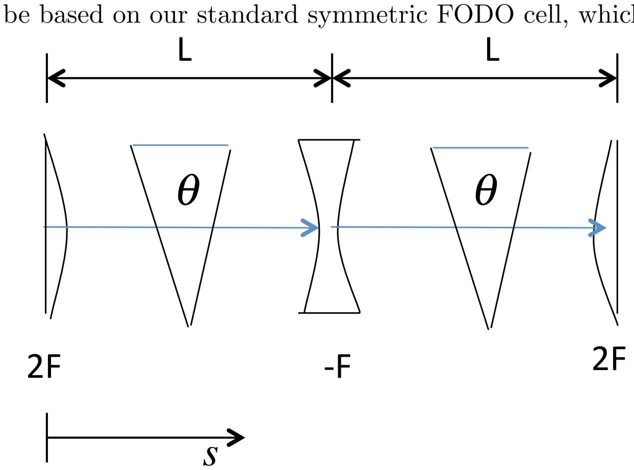
January 30, 2014

This copy of the final includes solutions, shown in red. Full derivations are given, along with example sources for the necessary equations. The requested answers are $\boxed{\text{boxed}}$. The number of possible points for each section of each problem is shown, with a total of 77 points being possible for the exam. Generally, if an error in one calculation results in errors in subsequent calculations, full credit will be given for the later calculations, provided their answers are consistent with the earlier (ie, incorrect) value.

General Guidelines

- This is an "open book" exam. You may use the text, lectures, homeworks, or any of the recommended resources, including last year's final. You are expected to work independently and to not seek out other sources for the solutions.
- There are a total of four problems, which do not have equal weight.
- You may use anything that appeared in the lectures, textbook or assigned homework, without re-deriving it.
- Full or partial credit will only be given if your reasoning can be followed, so show your work. Please give answers in the requested [units] when specified.
- The exam is due at 9AM tomorrow. Late exams will have their score reduced by 10%, with an additional 10% deducted for each additional hour.
- All problems are straightforward applications of what you have learned. There are no trick questions or complex calculations. If you find yourself working hard, it's a good sign you're not doing the problem correctly.
- If you think there's a problem with the test, contact Eric at 630-336-1893 or prebys@fnal.gov. Any necessary corrrections or clarifications will be sent to the email list and posted on the web page, so check both frequently!

The first three problems will be based on our standard symmetric FODO cell, which we have seen many times:



where each cell contains focusing and defocusing quadrupoles of focal lengths (in the horizonal plane) of F and $-F$, respectively, spaced L apart. The entire length between the quadrupoles is taken up by identical bend dipoles, each of which bends the beam by an angle θ. Positions s within the cell are measured from the center of the first focusing quadrupole. You may use the thin lens approximation for the quadrupoles, assume each (half period) dipole is $\approx L$ long, ignore any fringe field effects of the dipoles, and assume there are no horizontal dipole fields.

Problem 1 (18 points total)

a. Write expressions for all Twiss parameters in the bend plane (α_x,β_x, and γ_x) at middle of the the first bend ($s = L/2$). Express your answers in terms of (as appropriate) β_{max}, β_{min}, F, and L. (8 points)

We know from HW 2.3 and 2.4 that at the beginning of the cell

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_{max} \\ \left(\frac{1}{\beta_{max}}\right) \end{pmatrix}$$

There are two equally valid ways to propagate these to $s = L/2$...

The first is to propagate them piecewise through the half lens and then half a drift, as shown in Lecture 4, page 5. Going through the first half lens changes the intial values to

$$\alpha_1 = \alpha_0 + \beta_0 \frac{1}{f} = \frac{\beta_{max}}{2F}$$

$$\beta_1 = \beta_0 = \beta_{max}$$

$$\gamma_1 = \gamma_0 \pm 2\frac{\alpha_0}{f} + \frac{\beta_0}{f^2} = \frac{1}{\beta_{max}} + \frac{\beta_{max}}{4F^2}$$

We then propate these through a drift of length $L/2$

$$\alpha(s = L/2) = \alpha_1 - \gamma_1 s = \boxed{\frac{\beta_{max}}{2F} - \left(\frac{1}{\beta_{max}} + \frac{\beta_{max}}{4F^2}\right)\frac{L}{2}}$$

$$\beta(s = L/2) = \beta_1 - 2\alpha_1 s + \gamma_1 s^2 = \boxed{\beta_{max} - \frac{\beta_{max}}{2F}L + \left(\frac{1}{\beta_{max}} + \frac{\beta_{max}}{4F^2}\right)\frac{L^2}{4}}$$

$$\gamma(s = L/2) = \gamma_1 = \boxed{\frac{1}{\beta_{max}} + \frac{\beta_{max}}{4F^2}}$$

An alternative procedure is to calculate the transfer matrix from the beginning of the cell (half lens+half drift)

$$M = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2F} & 1 \end{pmatrix} = \begin{pmatrix} (1 - \frac{L}{4F}) & \frac{L}{2} \\ -\frac{1}{2F} & 1 \end{pmatrix}$$

We can then use the general transformation matrix (Lecture 4, page 4)

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} (m_{11}m_{22} - m_{12}m_{21}) & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}$$

Because $\alpha_0 = 0$, we can ignore the first column of the matrix, so this becomes

$$\begin{pmatrix} \alpha(s = L/2) \\ \beta(s = L/2) \\ \gamma(s = L/2) \end{pmatrix} = \begin{pmatrix} (...) & (\frac{1}{2F} - \frac{L}{8F^2}) & -\frac{L}{2} \\ (...) & (1 - \frac{L}{4F})^2 & \frac{L^2}{4} \\ (...) & \frac{1}{4F^2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_{max} \\ \left(\frac{1}{\beta_{max}}\right) \end{pmatrix}$$

so we have

$$\alpha(s = L/2) = \boxed{\left(\frac{1}{2F} - \frac{L}{8F^2}\right)\beta_{max} - \frac{L}{2\beta_{max}}}$$

$$\beta(s = L/2) = \boxed{\left(1 - \frac{L}{4F}\right)^2 \beta_{max} + \frac{L^2}{4\beta_{max}}}$$

$$\gamma(s = L/2) = \boxed{\frac{\beta_{max}}{4F^2} + \frac{1}{\beta_{max}}}$$

We see that with a slight algebraic rearrangement this is the same answer, as it must be.

b. If I want the phase advance across one cell to be *exactly* 60°, what value of F should I use, in terms of L? Simplify your answer as much as possible. (2 points)

From HW 2.3, we have

$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

so

$$\sin 30° = .5 = \frac{L}{2F} \rightarrow \boxed{F = L}$$

c. I build a ring out of N_{cell} of these identical FODO cells. What is the horizontal tune ν_x of the machine? (2 points)

The tune is related to the total phase advance, so

$$\nu_x = \frac{N_{cell}\mu}{2\pi} = \frac{N_{cell}(60)}{(360)} = \boxed{\frac{1}{6}N_{cell}}$$

d. Write an expression for the bend angle θ per (half cell) dipole in terms of N_{cell}. (2 points)

We have to make a complete circle, with each cell bending the beam by 2θ, so

$$N_{cell} \times 2\theta = 2\pi \rightarrow \theta = \frac{2\pi}{2N_{cell}} = \boxed{\frac{\pi}{N_{cell}}}$$

e. Assuming that my ring contains small anomalous magnetic errors up to and including sextupole terms, what are the values of the fractional tune I should avoid? (2 points)

We showed (Lecture 11, page 10) that quadrupoles can lead to resonances at the whole and half integer tunes and that sextupoles can lead to resonances at third integer tunes, so we should avoid tunes with a fractional component of

$$\boxed{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \text{ and } 1}$$

f. In light of this, and my 60° phase advance, write an expression or expressions for the *allowed* values of N_{cell}. (2 points)

We showed above that the tune is given by

$$\frac{1}{6}N_{cell}$$

so the fractional tune will be a multiple of 1/6. The only values that don't fall on the resonances in the previous part are 1/6 and 5/6, so the allowed values of N_{cell} are

$$\boxed{N_{cell} = 6m + 1 \text{ or } 6m + 5}$$

where m is an integer

Problem 2 (23 points total)

Now let's put in some numbers. I design a ring to accelerate protons from kinetic energy of $K_{inj} = 2$GeV to $K_{ext} = 20$GeV. I make each FODO cell a *total* of 30m long. Assume I still want a phase advance of $\mu = 60°$ per cell.

a. What are the values for (relativistic) β, momentum p[GeV/c], and beam rigidity $(B\rho)$[T-m] at injection and extraction? (5 points)

We can build a table with the parameters we will need now and later. The requested values are shown boxed.

Value	**Formula**	**Injection**	**Extraction**
K [GeV]	-	2	20
m [GeV/c^2]	-	.935	.935
E [GeV]	$K + mc^2$	2.935	20.935
p [GeV/c]	$\sqrt{E^2 - (mc^2)^2}/c$	$\boxed{2.78}$	$\boxed{20.92}$
β	pc/E	$\boxed{.948}$	$\boxed{.999}$
γ	$E/(mc^2)$	3.13	22.32
$(B\rho)$ [T-m]	p[GeV]/.300	$\boxed{9.28}$	$\boxed{69.72}$

b. What value of focal length F[m] do I need? (2 points)

In 1(b), we showed that for a 60° phase advance, $F = L$. Since L is half the total length of the cell

$$F = L = 30/2 = \boxed{15\text{m}}$$

c. If I use 2m long quadrupoles for the focusing magnets, what is the required gradient B'[T/m] when the beam is at the highest energy? (2 points)

The relationship betwen the gradient and the focal length is given by (Lecture 3, page 3)

$$f = \frac{(B\rho)}{B'l}$$

From the 20 GeV column above, we have $(B\rho) = 69.72$ T-m, so

$$B' = \frac{B\rho}{Fl} = \frac{(69.72)}{(15)(2)} = \boxed{2.32\text{T/m}}$$

d. I choose the most stable value of N_{cell} between 26 and 30. Based on your answer to 1(f), what is it? (2 points)

The only value which satisfies my criteria in 1(f) is

$$4 * 6 + 5 = \boxed{29}$$

e. What is the numerical value for the horizontal tune ν_x? (1 point)

From 1(c), we have

$$\nu_x = \frac{N_{cell}}{6} = \frac{29}{6} = \boxed{4.833}$$

f. Give numerical values for maximum values of the betatron function β[m] and and the dispersion D[m]. (4 points)

From HW 2.4 (S&E 3.12), we have

$$\beta_{max} = 2L\frac{1 + \sin\frac{\mu}{2}}{\sin\mu} = 2(15)\frac{1 + \sin 30°}{\sin 60°} = 2(15)\frac{(1 + .5)}{\sqrt{3}/2} = \boxed{51.96\text{ m}}$$

From HW 3.2, we have

$$D_{max} = \theta L \frac{1 + \frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}$$

We get the value of θ from 1(d) and write

$$D_{max} = \left(\frac{\pi}{N_{cell}}\right) L \frac{1 + \frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}} = \left(\frac{\pi}{29}\right)(15)\frac{1 + \frac{1}{2}(.5)}{(.5)^2} = \boxed{8.12 \text{ m}}$$

g. If my (95%) normalized beam emittance is $\epsilon_{95} = 10\pi$-mm-mr, what is the maximum RMS beam size in x, (σ_x) [mm] during injection, assuming for the moment that there is negligible momentum spread? (3 points)

The relationship between the *normalize* emittance and the RMS beam size is given by (Lecture 4, pages 12-13)

$$\sigma_x = \sqrt{\frac{\epsilon_{95}\beta_{max}}{6\beta\gamma}} = \sqrt{\frac{(.000010)(51.96)}{6(.948)(3.13)}} = .0054 \text{ m} = \boxed{5.4 \text{ mm}}$$

h. Let's assume that at at injection, I want the RMS contribution to the beam size due to the momentum spread at the point of maximum dispersion to be roughly the same as the contribution to the beam size due to the betatron oscillations in the previous part. What's the maximum allowable RMS momentum spread $\Delta p/p_0 = \delta$ I can have? What does this correspond to in terms of energy distribution σ_E[MeV]? (4 points)

The RMS due to dispersion is given by (lecture 5, page 2)

$$\sigma = D\delta$$

So to satisfy this condition, we have

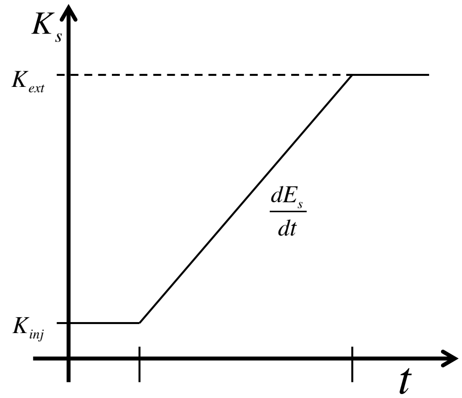
$$\delta = \frac{\sigma_x}{D_{max}} = \frac{(.0054)}{(8.12)} = \boxed{.00066}$$

From Lecture 2, page 6 (proven in HW 1.4), we have the relationship between the momentum spread and the energy spread

$$\frac{\Delta p}{p} = \delta = \frac{1}{\beta^2}\frac{\Delta E}{E} \rightarrow \sigma_E = E\beta^2\delta = (2.938)(.948)^2(.00066) = .0018 \text{ GeV} = \boxed{1.8 \text{ MeV}}$$

Problem 3 (22 points total)

Moving on to the longitudinal plane... Start with the synchrotron in the previous problem. Assume that we are going to inject at a kinetic energy $K_{inj} = 2$GeV, accelerate at a linear rate to $K_{ext} = 20$GeV, stop accelerating and extract it; that is, assume we are not accelerating at the time beam is injected and extracted.



a. If I design an RF system with a harmonic of $h = 120$, what are the injection and extraction frequencies f_{inj} and f_{ext}[MHz]. (3 points)

The frequency of the RF system is given by

$$f = h\frac{\text{velocity}}{\text{circumference}} = h\frac{\beta c}{2N_{cell}L}$$

Taking these values from Problem 2, we have

$$f_{inj} = (120)\frac{(.948)(3\times 10^8)}{2(29)} = 39.2\times 10^6 \text{ Hz} = \boxed{39.2 \text{ MHz}}$$

$$f_{ext} = (120)\frac{(.999)(3\times 10^8)}{2(29)} = 41.3\times 10^6 \text{ Hz} = \boxed{41.3 \text{ MHz}}$$

b. What are the injection and extraction slip factors η_{inj} and η_{ext}? (you may assume that $\gamma_t \approx \nu_x$) (2 points)

The slip factor is given by (Lecture 5, page 7)

$$\eta = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)$$

We get the tune from 2(e) and γ from the kinematic factors in 1(a), so

$$\eta_{inj} = \left(\frac{1}{(4.833)^2} - \frac{1}{(3.13)^2}\right) = \boxed{-.059}$$

$$\eta_{ext} = \left(\frac{1}{(4.833)^2} - \frac{1}{(22.32)^2}\right) = \boxed{.040}$$

c. I want to have at least a factor of 4 overhead in the bucket height ΔE_b over the RMS energy distribution of the beam at injection; that is $\Delta E_b = 4\sigma_E$. What value of the longitudinal beta function β_L [s/eV] does my RF system need to have if my energy spread just satisfies the condition specified at the end of problem 2? (3 points)

The bucket height is given by (Lecture 8, page 12)

$$\Delta E_b = 2\frac{\sqrt{1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s}}{\omega_{RF}\beta_L}$$

at injection, $\phi_s = 0$ so

$$\beta_L = \frac{2}{\omega_{RF}\Delta E_b} = \frac{2}{2\pi f_{RF}(4\sigma_E)} = \frac{2}{2\pi(39.2\times10^6)(4)(1.8\times10^6)} = \boxed{1.16\times10^{-15}\text{ s/eV}}$$

d. If this is my value of β_L, calculate the following *at injection*:

- The synchrotron tune ν_s (3 points)

  The relationship between the tune and the longitudinal beta function is given by (Lecture 8, page 8)

  $$\beta_L = \frac{\tau|\eta|}{2\pi E_s\beta^2\nu_s} \to \nu_s = \frac{\tau|\eta|}{2\pi E_s\beta^2\beta_L}$$

  The period τ is given by

  $$\tau = \frac{2N_{cell}L}{\beta c} = \frac{2(29)(15)}{(.948)(3\times10^8)} = 3.06\times10^{-6}\text{ s}$$

  so the synchrotron tune is

  $$\nu_s = \frac{(3.06\times10^{-6})(.059)}{2\pi(2.934\times10^9)(.948)^2(1.16\times10^{-15})} = \boxed{.0094}$$

- The total required RF voltage V_0 [MV] (3 points)

  We can calculate this from the tune with (Lecture 8, page 8)

  $$\beta_L = \sqrt{-\frac{\tau\eta}{eV_0\omega_{RF}E_s\beta^2\cos\phi_s}} \to V_0 = -\frac{\tau\eta}{e\beta_L^2(2\pi f_{RF})E_s\beta^2\cos\phi_s}$$

  so for $\phi_s = 0$, we have

  $$V_0 = -\frac{(3.06\times10^{-6})(-.059)}{e(1.16\times10^{-15})^2(2\pi)(39.2\times10^6)(2.94\times10^9)(.948)^2} = 2.1\times10^5\text{ V} = \boxed{.21\text{ MV}}$$

e. I advance the accelerating phase to begin accelerating the beam. If I can set my synchronous phase ϕ_s in 5° increments (5°, 10°, 15°, etc), what is the maximum value for $\phi_s[^\circ]$ which will keep my bucket overhead *at least* a factor of 3 during initial acceleration? This problem must be solved numerically. (Hint: you've already solved the equation for the case of no acceleration. Isolate the dependence on ϕ_s and plug in increasing values.) (5 points)

Our equation for the bucket height is

$$\Delta E_b = 2\frac{\sqrt{1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s}}{\omega_{RF}\beta_L}$$

and the equation for β_L is

$$\beta_L = \sqrt{-\frac{\tau\eta}{eV_0\omega_{RF}E_s\beta^2\cos\phi_s}}$$

so in terms of ϕ_s

$$\Delta E_b \propto \sqrt{\left(1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s\right)\cos\phi_s} = \sqrt{\cos\phi_s-\left(\frac{\pi}{2}-\phi_s\right)\sin\phi_s}$$

We know that at $\phi_s = 0$, this gave a factor of 4 bucket overhead, and we want to keep a factor of 3 bucket overhead, so we want to find the largest value of ϕ_s for which

$$\sqrt{\cos\phi_s-\left(\frac{\pi}{2}-\phi_s\right)\sin\phi_s} \equiv f(\phi_s) > \frac{3}{4}$$

We plug in values of ϕ_s in 5° increments

ϕ_s	$f(\phi_s)$
0	1
5	.93
10	.86
15	.79
20	.72

So if we want to keep our bucket overhead at least a factor of 3, our maximum synchronous phase angle is

$$\phi_s = \boxed{15^\circ}$$

Note that $\phi_s = 20^\circ$ is actually closer, but the problem clearly states that we want *at least* a factor of 3.

f. Given these values of V_0 and ϕ_s, what is initial acceleration ramp dE_s/dt[GeV/s]? (3 points)

The energy gained each turn will be

$$\Delta E = eV_0 \sin\phi_s$$

so the rate of acceleration is

$$\frac{dE}{dt} = \frac{eV_0 \sin\phi_s}{\tau} = \frac{e(.21\times 10^5)(\sin 15^\circ)}{(3.06\times 10^{-6})} = 1.76\times 10^{10}\ \text{eV/s} = \boxed{17.6\ \text{GeV/s}}$$

Problem 4 (14 points total)

Consider a proton-proton collider with the following parameters:

- Beam energy: E_b $(\gg m_p c^2)$
- Circumference: C
- Number of bunches: n_b
- Number of protons ber bunch: N_b
- Normalized RMS emittance: $\epsilon_x = \epsilon_y = \epsilon_N$.
- A single, low beta interaction region.
- Final focusing triplets are located a distance d away from the interaction point.

Assume that we are running at the maximum luminosity L_0, limited by:

- A fixed normalized beam emittance ϵ_N.
- The beam-beam tune shift ξ.
- The aperture A of the final focusing quads ($A \equiv$ diameter of the beam pipe)
- The minimum bunch spacing; that is, n_b is a maximum.

In terms of the above parameters, answer the following questions:

a. Write an expression for the betatron function β_A in the final focusing triplet, assuming I want at least a 6σ clearance for the beam (assume $\beta_x \approx \beta_y$ at this point)? (5 points)

We have that the beam size is given by

$$\sigma = \sqrt{\frac{\beta_A \epsilon_N}{\beta\gamma}}$$

We want this to be 1/6 of the distance to the aperture, but remember A is a *diameter*. Since beam size is measured from the center of the beam pipe, we must divide it by 2.

$$\sigma = \frac{1}{6}\left(\frac{A}{2}\right) = \sqrt{\frac{\beta_A \epsilon_N}{\beta\gamma}} \to \beta_A = \left(\frac{A}{12}\right)^2 \frac{\beta\gamma}{\epsilon_N} \approx \boxed{\left(\frac{A}{12}\right)^2 \frac{1}{\epsilon_N}\left(\frac{E_b}{mc^2}\right)}$$

where we have used $\beta \approx 1$ and $\gamma = E_b/(mc^2)$.

b. What is the approximate minumum value for the beta function β^* at the collision point? You may assume $\beta^* \ll d$. (Hint: How does the beta function behave as I move away from the minimum at the collision point?) (5 points)

As we move away from the minimum beta, our beta increases as (Lecture 7 page 11)

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*} \to \beta_A = \beta^* + \frac{d^2}{\beta^*} \approx \frac{d^2}{\beta^*}$$

where we have used the assumption that $d \gg \beta^*$. So we can write β^* in terms of β_A as

$$\beta^* = \frac{d^2}{\beta_A} = \boxed{d^2 \epsilon_N \left(\frac{12}{A}\right)^2 \left(\frac{mc^2}{E_b}\right)}$$

c. If I double the energy, but keep the circumference C the same (that is, I double the strength of my magnets), by what factor will the maximum luminosity increase, assuming I can change β^*, but that I am subject to all the same limitations itemized above. (4 points)

Our expression for the tuneshift limited luminosity is (Lecture 14, page 10)

$$L = f\frac{n_b N_b^2 \gamma}{4\pi\beta^* \epsilon_N}$$

Because of the γ factor in the numerator, simply doubling the energy and keeping everything else the same will double the luminosity, but what else can we do? Since we are tuneshift limited, we cannot increase (N_b/ϵ_N). Since we said ϵ_N was fixed, this means N_b is also fixed. We also said n_b is a maximum. However, we see from previous problem that doubling the energy will allow us to reduce β^* by a factor of 2, while still maintaining my aperture clearance at the final focus. This means that

$$L_{max}(E = 2E_b) = f\frac{n_b N_b^2 \gamma}{4\pi\beta^* \epsilon_N} = f\frac{n_b N_b^2 (2\gamma_0)}{4\pi(\beta_0^*/2)\epsilon_N} = \boxed{4L_0}$$